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*PRODUCTION OF ILLUDIN M AND OF A FOURTH CRYSTALLINE COMPOUND BY CLITOCYBE ILLUDENS**

By Marjorie Anchel, Annette Hervey and William J. Robbins

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Communicated September 15, 1952

The isolation of three crystalline compounds from culture liquids of *Clitocybe illudens*, strain 72027-S, was recently reported.[1] Of the two antibacterial compounds, illudin M was obtained in relatively small amounts. A strain 14610-S has been found[2] which is more efficient under our conditions of culture than 72027-S for the production of illudin M. It produces larger amounts of this compound, and also produces relatively small amounts of illudin S. Isolation of illudin M consequently is simplified. A fourth compound, not antibacterial in our tests, has been isolated from culture liquids of this strain.

Clitocybe illudens (14610-S) was grown at 25° C. in 2800 ml. Fernbach flasks containing beechwood shavings and a corn-steep medium.[3] In about five weeks, at which time the mycelial mat completely covered the surface of the liquid, the activity against *Staph. aureus* was usually 32 dilution units[4] per ml. and against *Myco. smegma* 512 to 1024 dilution units per ml. Reflooding the mats with fresh corn-steep medium produced liquids with as high activity in about three weeks. Liquids tested three months after reflooding had retained their antibacterial activity; therefore, the mats were usually reflooded at convenient intervals of from 3 to 6 weeks. Mats kept for as long as 15 months continued to produce active culture liquids after being reflooded thirteen times.

The isolation procedure consisted of extraction of the culture liquid with chloroform (three extractions, 20, 10, and 10 per cent of the volume), and concentration of the chloroform extract *in vacuo*, to a heavy syrup. On standing, this deposited crystals which consisted of a mixture of illudin M and a new acidic compound. These were filtered off, washed with a solution of chloroform-hexane, and dried *in vacuo*. Separation of the two compounds was accomplished by sublimation, which left behind the acidic

compound, or by extraction of a solution of the compounds in ether with dilute bicarbonate solution, which left behind the illudin M. When obtained by the latter method, illudin M was best purified further by sublimation, which removed any traces of illudin S. (Most of the illudin S remained in the culture liquid after chloroform extraction.) If the extracted culture liquid was treated with Norit A according to the procedure previously described,[1] illudin S was obtained as crude crystals from the ethyl acetate extract of the concentrated aqueous acetone eluate. In five batches the yield varied from 0.04 to 0.17 g. per liter of culture liquid and averaged 0.08 g. per liter. The yield of sublimed illudin M was about 0.22 g. per liter, and represented 25 to 60 per cent of that present in the culture liquid as calculated from the activity of the latter. The ratio of the yields of illudin M to illudin S was: 1.29:1 for this strain as compared to 0.12:1 for strain 72027-S. The yield of the inactive acidic compound was about 0.05 g. per liter of culture liquid.

The acidic compound crystallized from 95 per cent ethanol in flat, prismatic rods. It melted at 216° (corr.). Analytical values agree with the molecular formula $C_{21}H_{22}O_7$, and with the presence of two acidic groups. Found:[5] C, 65.07; H, 5.88; N.E., 190: M.W., 365. Calc. for $C_{21}H_{22}O_7$ (386.30): C, 65.27; H, 5.74; N.E. 193; M.W., 386. The ultra-violet absorption spectrum showed maxima at 247 mμ (ϵ = 43,900) and at 330 mμ (ϵ = 2900) indicating the presence of an α,β-unsaturated carbonyl group.

* This investigation was supported in part by a research grant from the National Microbiological Institute of the National Institutes of Health, Public Health Service. The authors wish to acknowledge the technical assistance of Marvin Cohen, Jane Bancroft and Alice Hesse.

[1] These PROCEEDINGS, **36**, 300–305 (1950).

[2] We are indebted to Ross W. Davidson for this strain.

[3] The medium contained per liter, 1 g. KH_2PO_4, 0.5 g. KCl, 3 g. $NaNO_3$, 0.5 g. $MgSO_4 \cdot 7H_2O$, 40 g. dextrose and 5 g. Staley special nutrient 114 (corn-steep).

[4] Anchel, M., *J. Biol. Chem.*, **177**, 169–177 (footnote 1) 1948.

[5] Microanalyses were performed by the Huffman Microanalytical Laboratories, Denver, Colo.

THE REACTION CONTROLLING FLORAL INITIATION

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Communicated September 9, 1952

Reproduction in many plants and animals is controlled by the length of the night and the cycle of day and night. A dark reaction affords the measure of time and this reaction can be quickly stopped or reversed by light such as that from the rising sun. The reaction is possibly universal in living forms but its more extreme and evident manifestation is recognized to a rather limited extent. Thus flowering of cosmos is determined by long nights while, to casual examination, growth of petunia and tomato is independent of night length. Similarly, reproduction in some species such as goats, turkeys and snails depends on the length of night while the cow and man seem unresponsive.

Control of morphogenesis serves better as evidence for the photoreactions in plants. Equivalence of the morphogenic and reproductive controls can be tested by the details of the action spectra for the photoreaction. Thus is established the equivalence of the control for the size of a pea leaf, after it has developed somewhat in darkness and then is exposed briefly to light, with the control of floral initiation in soybean and in barley plants.[1] Control of bulbing in the onion or color of coat in the varying hare could be followed in a similar way.

An understanding of the control reaction, in contradistinction to establishing the equivalence of varied phenomena, came from the study of germination response of lettuce seed to red and near infra-red radiation.[2]

The reversible photoreaction and an accompanying dark reaction demonstrated for lettuce seed and extended in this study to control of floral initiation follow:

(1) Pigment + RX $\underset{\text{Infra Red}}{\overset{\text{Red}}{\rightleftharpoons}}$ Pigment X + R

6500 Max. — 7300 Max.
Floral long night plants — Vegetative long night plants
Dormant lettuce seed — Germinating lettuce seed

(2) Pigment X + R $\underset{\text{Darkness}}{\overset{}{\rightleftharpoons}}$ Pigment + RX

(3) with (1) Substrate + RX $\xrightarrow{\text{Darkness}}$ R + Substrate X

(4) with (2) Substrate X + R $\longrightarrow$ RX + Substrate

The nature of the pigments and the reactants RX and R are still to be established.

The purpose here is to present evidence that these reactions also control floral initiation of cocklebur (*Xanthium saccharatum* Wallr.), which is selected as representative of photoperiodically responsive plants. Inadequacies in previous work, chiefly by us, on action spectra for floral initiation have to be examined.

Radiation in the red when used to interrupt a dark period has been found to be the most effective portion of the spectrum for control of flowering for both long- and short-night plants. The maximum effectiveness is in the region of 6300 to 6600 Å which is also the region for maximum promotion of germination of lettuce seed. Radiation in the region of 7000 to 7600 Å hitherto has not been recognized as effective in control of flowering, but this region has been shown to be strongly inhibitory for germination of lettuce. This infra-red effect on seed germination gave us pause to think about the way in which the action spectra were measured for floral initiation. In all cases control was effected by a brief light interruption near the middle of each long dark period.

Description of an actual experiment with cocklebur is instructive. All leaves except one are removed from young plants. The plants are then placed on 12-hour dark periods and 12-hour light periods, which are adequate for floral initiation if used for one or more cycles. Usually two or three such cycles are used and the dark periods are interrupted with radiation in various wave-length bands between 4000 and 10,000 Å. The radiation is supplied with a two-prism spectrograph adequate in dispersion to irradiate a single leaf with a wave-length band of about 100 Å in the region of 5000 Å. After the dark-period interruption treatments are completed the plants are returned to 8-hour dark periods and 16-hour light periods to continue development. Whether the terminal meristem is floral or vegetative is ascertained 12 days later by dissection. Under these conditions cocklebur plants remain vegetative if not previously induced by long nights. The minimum energy required various wave-length regions for *suppression of floral initiation* is found by varying the irradiance. These measurements lead to the action spectra.

Radiation at wave-lengths greater than 7200 Å was found to be ineffective in suppression of floral initiation of cocklebur. There are two possible conclusions from such a result—namely: (1) radiation at wave-lengths greater than 7200 Å has no effect on floral initiation, and (2) it has the same effect as darkness. We were aware that the dark-period reaction was the one effective in measurement of time, but the second possibility did not occur to us.

If the cocklebur plants are irradiated, as described, first with unfiltered radiation from an incandescent-filament lamp adequate to prevent floral initiation as shown by controls and then on the spectrograph with radiation in the region, 7215 to 7450 Å, floral initiation occurs. The results

parallel those for lettuce seed germination. Radiation in the region of 7300 Å thus is effective in controlling flowering of cocklebur but its action is opposite to that of radiation at 6500 Å. Radiation in the region of 7300 Å promotes flowering and thus has the same effect immediately on the photoreaction as does a prolonged dark period.

We have now measured the action spectrum in terms of incident energy for *promotion* of floral initiation of cocklebur as described in the preceding paragraph. One set of results is given in table 1. The maximum response is in the region of 7215 to 7450 Å, which is closely equivalent to the results for lettuce seed germination. The same controlling reaction is operating.

The previously measured action spectra for regulation of floral initiation

TABLE 1

EFFECT OF VARIOUS WAVE-LENGTHS OF RADIATION ON PROMOTION OF FLOWERING OF *Xanthium saccharatum* PLANTS THAT HAD PREVIOUSLY RECEIVED RADIATION INHIBITORY TO FLOWERING

WAVE-LENGTH, Å	INCIDENT POWER (ERGS × 10^3/CM.^{2}/MIN.)	NUMBER OF FLOWERING PLANTS PER LOT OF 4 AFTER IRRADIATION FOR INDICATED TIME			
		16 MIN.	8 MIN.	4 MIN.	2 MIN.
6820–7020	87	0	0	0	0
7020–7215	93	1	3	0	0
7215–7450	99	3	3	3	1
7450–7700	102	1	1	1	0
7700–8010	105	0	0	0	0
8010–8300	111	0	0	0	0

TABLE 2

ENERGY REQUIRED TO CONTROL FLORAL INITIATION OF *Xanthium saccharatum* NEAR THE MIDDLE OF A 12-HOUR DARK PERIOD

REGION OF IRRADIATION, Å	EFFECT ON FLOWERING	ENERGY (KILOERGS/CM.^{2})	
		EXPT. 12	EXPT. 16
6330–6560	Prevention	36	400
7020–7450[a]	Promotion	300	100

[a] Previously irradiated adequately to prevent floral initiation as shown by controls.

thus depend upon the integrated absorptions of the two compounds, Pigment and Pigment X. They should be reexamined from this point of view. The various spectra while closely similar are not identical. In the red portion of the spectrum appreciable displacements from the action spectrum for cocklebur are evident as might result from a different poising of the photoreaction: The maximum effectiveness for each action spectrum is in the region of 6300 to 6600 Å, which is probably near the absorption maximum for one pigment. The curves expressing incident energy required for a given response of each species are parallel in the region of 5000 to 6000 Å but that for lengthening of a dark grown pea leaf[1] differs by about 600 Å from that for suppression of floral initiation in soybean.

Effectiveness of red radiation in promoting germination of lettuce seed decreases after they have been imbibed with water in excess of 20 hours while infra-red radiation correspondingly becomes more effective in suppressing germination. This is one of the several evidences for the coupling of the two responses. Similarly for floral initiation of cocklebur, as sensitivity to inhibition by red radiation increases that to promotion by infra-red decreases (table 2). The cause for this variation is still being examined; it might be the age of leaf irradiated or some pecularity of treatment. Previous work with soybeans[3] indicated that the third compound leaf on plants that had expanded from compound leaves was the most effective one for control of floral initiation. Irrespective of cause the reciprocal variation is pronounced in this and other experiments.

TABLE 3

DEPENDENCE OF FLOWERING RESPONSE OF *Xanthium saccharatum* ON THE TYPE OF IRRADIATION DURING THE LAST THIRTY MINUTES OF THE LIGHT PERIOD

LENGTH OF DARK PERIOD, HOURS	NUMBER OF FLOWERING PLANTS[a] PER LOT OF 4 AFTER IRRADIATION AS INDICATED		
	RED	CONTROL	INFRA-RED
7.0	..	..	4
7.5	..	..	4
8.0	0	0	4
8.5	0	4	4
9.0	4	4	..
9.5	4	4	..
10.0	4	4	..

[a] All except two leaves were removed from plants used in this experiment. The fluorescent source for red radiation gave an illumination of 100 f.-c. at the leaf surface before passing through the red cellophane. Infra-red was obtained by filtering direct sunlight through a Corning red purple ultra filter. The control plants received radiation from an a. c. carbon arc supplemented with filament lamps, with a total illumination at the leaf surface of about 200 foot-candles.

Others who might wish to extend these observations and do not have the rather unusual spectrographic equipment can perform many experiments with radiation restricted to broad regions by choice of sources and filters. The effective red region relatively free of infra-red radiation is given by light from a 4500° white fluorescent lamp filtered by several layers of red cellophane. The effective infra-red region is rather precisely separated from the red region by use of a 5.0 mm.-thick Corning red purple ultra filter. An incandescent-filament lamp, the crater of a carbon arc, or the sun can be used as a source of radiation depending upon the intensity required.

Filtered radiation was used, as mentioned in the preceding paragraph, to perform an experiment in which the critical length of the dark period required for floral initiation of cocklebur was changed. The light period

for a group of vegetative cocklebur plants was extended for 30 minutes with either red or infra-red radiation. Various lots of these plants were then given dark periods of seven different durations from 7 to 10 hours at 30-minute intervals. Control lots were subjected to the same dark periods as those irradiated in the red. Red radiation applied at the end of the light period lengthened the critical dark period from 8.5 hours to 9 hours while infra-red shortened the critical dark period to less than 7 hours (table 3).

The experiment can best be understood in terms of the reaction given earlier in the article. During the light period the reaction is displaced to the right and in darkness it goes thermally to the left. The rate of this thermal reaction determines the effectiveness of the dark period. During the light period plants are usually subjected to both red and infrared radiation. The proposed reversible photoreaction is at neither extreme although closest to that given by red radiation. Thus red light at the start of the dark period displaces the reaction to the right, thereby lengthening the required dark period. Infra-red radiation at the beginning of the dark period immediately displaces the reaction to the left, and thus shortens the dark period required for floral initiation. Again, the experiment is one of the many evidences for coupling of the two reactions.

We have worked much with effects of radiation between 4000 and 5400 Å in the violet, blue, and green on these types of phenomena. Further experiments are required, however, in this region in view of these more recent advances to understand clearly the characteristics of the electronic transitions in the pigment molecules involved, and to untangle the absorptions of the two pigments.

Attention should be called not only to the two pigments but also to the reactants R and RX. All four might be expected to vary in plants as will be observed in time. In this variation is to be found the explanation for the distinctive photoperiodic responses of *Brassica rapa L.* and other Cruciferae[4] to blue and red radiations.[5] Will here also be found part of the explanation for dormancy of buds, biennial character, vernalization, "ripeness to flower," and many other phenomena of plant growth? Where will it lead with animals?

[1] Parker, M. W., Hendricks, S. B., Borthwick, H. A., and Went, F. W., *Amer. J. Bot.*, **36,** 194 (1949).

[2] Borthwick, H. A., Hendricks, S. B., Parker, M. W., Toole, E. H., and Toole, Vivian K., PROC. NAT. ACAD. SCI., **38,** 662–666 (1952).

[3] Borthwick, H. A., and Parker, M. W., *Bot. Gaz.*, **101,** 806 (1940).

[4] Funke, G. L., *Vernalization and Photoperiodism*, Lotsya 1, Waltham, Mass., 1948, pp. 79–82.

[5] Growth responses to infra-red radiation by *Cosmos bipinnatus* Car., *Spinacia oleracea* L. var. Nobel, *Brassica alba* L. Boiss. and *Pisum sativum* L. have been described to us in private communication by Prof. E. C. Wassink and Dr. J. A. J. Stolwijk (Wageninn-

gen). These results are in course of publication. Some less striking effects of infra-red radiation on growth of Perilla have been reported by B. S. Moskov, *Akad. Nauk., Doklady* **71**, 391 (1950).

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*ELECTRON EXCHANGE-POLYMERS**

By Harold G. Cassidy

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Read before the Academy November 5, 1951; received September 26, 1952

It was proposed in an earlier publication,[1] that it should be possible to prepare polymers which, in partial analogy to the ion exchange resins which are proton-exchangers, would be capable of electron-exchange. Such substances were prepared,[2] and although they were of low molecular weight,[3] nevertheless they showed some of the expected properties. This opened up the concept of a whole new family of polymers, the redox polymers, which can be used to conduct oxidation and reduction reactions by the same methods employed with the familiar ion exchange-resins. High molecular weight substances of this nature have been prepared and studied in electrometric titrations, and in countercurrent applications.

Vinylhydroquinone was prepared by reactions described previously.[2] Since the hydroquinone portion of the molecule inhibited the vinyl polymerization of the other group, leading instead to low-molecular weight products, it was necessary to protect the hydroxyls. This was done satisfactorily by acetylation or benzoylation. The resulting monomers, vinylhydroquinone diacetate and vinylhydroquinone dibenzoate, were also new substances, and their structures were established. When either of these substances was heated in toluene with 1 mol. % benzoyl peroxide smooth polymerization occurred with the production of high polymers. In one preparation, for example, a polyvinylhydroquinone dibenzoate with a number average molecular weight of 52,000 (*ca.* 160 monomer units) was prepared. It showed a saponification equivalent of 180; that calculated for the ester is 172. Copolymers were also prepared. Thus a 1 : 1 molar mixture of vinylhydroquinone dibenzoate with styrene, as 20% solution in toluene, with benzoyl peroxide catalyst yielded, on heating at 85° for 72 hours a polymer with a number average molecular weight of 41,300 (about 200 monomer units on a 1 : 1 basis). It showed a saponification equivalent of 242. That calculated on a 1 : 1 basis is 224.

The ester groups were subsequently removed by saponification with sodium ethylate solution in the absence of oxygen. The resulting free hydroquinone polymers and copolymers were obtained following purification and freeze-drying as fluffy white powders. They tended, with one

exception, to become pink on standing. This color change was apparently due to oxidation; the reason for the exception is not known.

The free polymers could be dissolved in 90% acetic acid, or in tertiary butyl alcohol to which a little water had been added. The presence of a very polar small molecule in the solvent seems essential to dissolution and suggests that the polymers are cross-linked through interactions (hydrogen bonds?) between their polar groups, and only when these are swamped with small polar molecules does the polymer become soluble.

A cross-linked polymer of vinylhydroquinone, styrene and divinyl benzene was prepared and was found to be insoluble in all solvents. Copolymers were prepared also of vinylhydroquinone with α-methyl styrene and with vinyl pyridine. That copolymers were indeed produced was suggested by solubility behavior, but has not yet been established by the necessary fractional precipitation procedures.

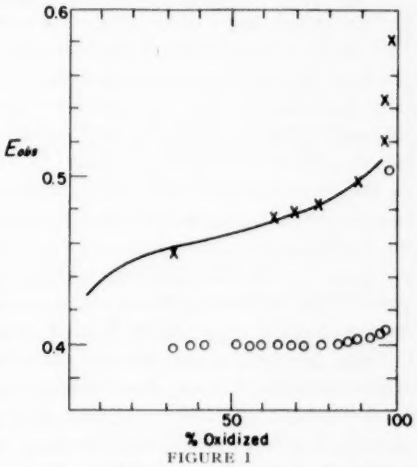

FIGURE 1

Lower curve: oxidative titration of polyvinyl hydroquinone in 90% acetic acid with bromine in 90% acetic acid. The polymer precipitated during the titration. Upper curve: titration of hydroquinone in 90% acetic acid with bromine in the same solvent. The solid line is the theoretical curve for a 2-electron titration.

The free hydroquinone polymers show the following interesting behaviors. When dissolved, and titrated in the dissolved or colloidal state, the behavior is much like conventional oxidizable substances, though the shapes of the curves imply a mixture of substances with slightly different normal potentials. If the titration is carried out with precipitation of the oxidized form (or perhaps of the quinhydrone intermediate) the curve takes the characteristic flattened shape[4] shown in figure 1. The titrations have to be carried out rather slowly, and in many cases stable potentials are obtained which imply equilibrium. In other cases difficulties are encountered in obtaining agreement between two electrodes in the same system. During the course of the titration, the polymer solution or suspension takes on a pink color which usually gives place at the end-point to a yellow or orange color. It appears that the pink color is due to quinhydrone formation or a related phenomenon, and that the fully oxidized polymer is yellow.

It was predicted at the time the first part of this paper was presented,* and has since been shown experimentally,[5] that when the linear polymer is

dispersed by precipitation onto a diatomaceous earth, so that a large surface is exposed, or when the cross-linked polymer is finely ground with the diatomaceous earth, electron-exchange columns can be prepared analogous to ion-exchange chromatographic columns. Such a column, prepared by dispersing 14.945 mg. (0.117 meq. on a 1 : 1 basis) of a copolymer of vinylhydroquinone and α-methyl styrene on 250 mg. purified filter-aid, showed the following behavior. The dispersed reduced polymer, in a chromatography tube, when oxidized by passing through it 0.0155 *N* iodine in KI, mixed with an equal volume of phosphate buffer of pH 7.8, reduced 0.122 meq. iodine. When the column was washed and treated with $2^1/_2$ % KI in 4 *N* HCl a zone of iodine formed and migrated down the column, constantly broadening. The iodide oxidized to iodine by the column was 0.117 meq. A repetition of the process showed 0.105 meq. iodine reduced, followed by 0.098 meq. iodide oxidized. When the column in the reduced form was washed free of iodine and iodide, and treated with 0.0179 *N* potassium dichromate to which $^1/_5$ of its volume of 8 *N* HCl was added, the column reduced 0.106 meq. dichromate. Similar cycles were repeated, but the column seemed to lose some capacity. Perhaps the repeated oxidation and reduction, with concomitant changes in shape and polarity of the molecules allowed them to coil up and expose fewer and fewer active groups to the reagent. Similar observations were made during electrometric titrations. The oxidized polymer could, under some conditions, be reduced, but on reoxidation invariably showed an increased equivalent weight.[3]

These new substances, the electron exchangers, or "redox resins," show properties which permit the following conclusions and predictions. The substances are reversibly oxidizable and reducible. The observed loss in capacity can certainly be prevented by methods already used in the ion-exchanger field. These substances, used as columns or beds through which reagents may be percolated, act in effect in the reducing cycle, like Jones reductors, with the important differences that they are more versatile, being convertible to oxidizers, and that *they do not contaminate the solution* except for effects on hydrogen ion concentration.

These features make these substances of considerable interest in applications where contamination, and especially metal ion contamination, is undesirable. A number of biochemists and enzymologists have expressed interest in these substances as agents for maintaining reducing conditions in a system without contaminating it with soluble substances. These resins may be of use in the field of water purification where, for example, the chromate ion (from plating operations) is not readily removed, but the chromic ion is.

It is to be expected that these substances will be able to effect oxidations of an irreversible type if there is present in the system some carrier redox substance. Such aided transfers of electrons suggest that these substances may serve as models for the study of enzyme-like behavior.

A polymer chain, or mass, of these substances in the quinhydrone (half-oxidized) state should have semiconducting properties, and should be interesting model substances from this view-point.

* Presented in large part before the Autumn meeting of the National Academy of Sciences at Yale University, November 5, 1951. This paper reports work done by Dr. Ivor H. Updegraff, Mr. Myer Ezrin and the author.

† Contribution No. 1126.

[1] Cassidy, H. G., *J. Am. Chem. Soc.*, **71**, 402 (1949).

[2] Updegraff, I. H., and Cassidy, H. G., *Ibid.*, **71**, 407 (1949).

[3] Ezrin, M., Updegraff, I. H., and Cassidy, H. G., *Ibid.*, submitted.

[4] Geake, A., and Lemon, J. T., *Trans. Faraday Soc.*, **34**, 1409 (1938).

[5] Cassidy, H. G., Ezrin, M., and Updegraff, I. H., *J. Am. Chem. Soc.*, submitted.

*THE THERMAL DECOMPOSITION OF CYCLOBUTANE**

BY FRITZ KERN AND W. D. WALTERS

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Communicated by W. A. Noyes, Jr., September 25, 1952

Introduction.—In an earlier publication[1] the homogeneous thermal decomposition of cyclobutane at 430–480°C. ($C_4H_8 \rightarrow 2C_2H_4$) has been reported to be a first-order reaction for initial pressures in the range 100–400 mm. Since the reaction was not inhibited by the addition of nitric oxide or propylene, which often markedly retard free radical chain reactions, it appeared that the decomposition might be a unimolecular reaction. In this event the investigation of the falling-off in the rate constant with a decrease in the initial pressure would be of interest. Therefore, the present study was undertaken in order to investigate the thermal decomposition of cyclobutane at lower pressures. In particular, the object of this work was to ascertain (*a*) whether the decomposition at low pressures is a simple reaction uncomplicated by side reactions and heterogeneous effects and (*b*) whether the first-order rate constants decrease as the initial pressure is lowered.

Experimental Procedure.—The cyclobutane used in these experiments was prepared by the photolysis of cyclopentanone[2] and was fractionated in a Podbielniak column. The purity of the material was tested in essentially the manner reported earlier.[1] Nitric oxide obtained from the Matheson Company was purified by a twofold distillation from −158° to −195°C. C. P. propylene, dimethyl ether, ethane and ethylene were supplied by the Ohio Chemical and Manufacturing Company. The Merck reagent-grade toluene was dried over anhydrous calcium sulfate. All of these compounds were degassed thoroughly before use.

The decomposition was carried out in a cylindrical Pyrex vessel of 490 ml. capacity. The heterogeneity of the reaction was tested in a vessel containing thin-walled Pyrex tubes which increased the surface-to-volume ratio by a factor of thirty-four.[3] The furnace and the temperature regulator were of the type previously used;[4] the temperature was measured by means of a standardized platinum, platinum–13% rhodium thermocouple connected to a Leeds and Northrup Type K-2 potentiometer. The error in the measured reaction temperature should not exceed ±0.5°C. Most of the pressure measurements down to 2 mm. were obtained by the use of a large-bore mercury manometer read with a cathetometer; pressures near 80 mm. were measured also with an ordinary capillary mercury manometer and scale. Under the same reaction conditions similar rates of decomposition were found with both devices. Pressures for the range 2 to 0.2 mm. were read with a small McLeod gauge.

The analysis of the products of the decomposition at low pressures was accomplished by the use of mass spectrometric measurements[5] and by absorption of the unsaturated hydrocarbons in activated sulfuric acid.[6] Analyses were performed not only on the final products but also on the reaction mixture at approximately the time of half decomposition. In the latter case the reaction mixture prior to analysis was separated into a more volatile fraction (vaporizable at −137°C.) and a less volatile fraction (vaporizable at room temperature).

Results and Discussion.—That the decomposition of one molecule of cyclobutane at low pressures yields two molecules of products was shown by the fact that in experiments in the unpacked vessel at 449°C. with initial pressures of 20, 10, 4 and 1 mm. and in an experiment in the packed vessel at 10 mm. the ratio of the final to the initial pressure was 2.00 ± 0.02 in every instance. Mass spectrometric analysis of the final products from an experiment at 459°C. and 5 mm. showed 99.7% ethylene. A secondary reaction of the ethylene does not appear to proceed at an appreciable rate at low pressures in the unpacked vessel. The non-condensable products in the cyclobutane decomposition, which were estimated from the pressure remaining after condensation at −195°C., do not exceed 0.2%.

Mass spectrometric measurements also showed that ethylene constitutes 99.8 and 99.9% of the more volatile fraction from experiments at 449°C. and 15 mm. pressure carried to about half completion in the unpacked and packed bulb, respectively. In the same experiments the less volatile fraction from the unpacked bulb was found to contain 98.5% cyclobutane and 1.0% ethylene, and that from the packed bulb, 98.3% cyclobutane and 0.9% ethylene. The trace constituents (0.1-0.2%) of the reaction mixture are more difficult to identify and determine, but hydrogen, propylene, butane, butadiene and hexane are among those reported in some of the analyses.

On the assumption that ethylene is the only product, the initial pressure and the pressure at the time of removal of the reaction mixture can be used to calculate the amount of ethylene (or fraction volatile at $-137°C.$) and cyclobutane (or fraction volatile at room temperature). A comparison of the calculated and the observed results is shown in table 1.

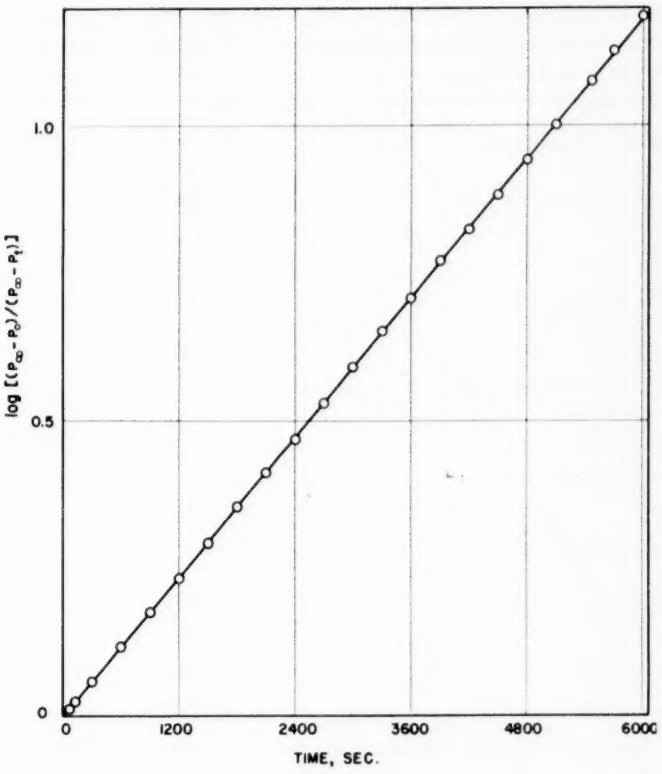

FIGURE 1

Decomposition of 10 mm. of cyclobutane at 449°C.

These observations indicate that significant side reactions do not occur in either the packed or the unpacked vessel. Moreover, the data show that the pressure measurements should be a reliable means of following the reaction.

With respect to the homogeneity of the decomposition it has been observed that at pressures of 10 mm. or above the reaction rate is practically unaltered by the increased surface of the packed bulb. In the 1-mm.

range, packing increases the rate by not more than 10% which in view of the 34-fold increase in surface-to-volume ratio indicates that the reaction in the unpacked bulb still proceeds almost entirely in the gas phase.

The kinetics of the low-pressure decomposition have been studied by the use of pressure measurements. In figure 1 the data from a 10-mm. experiment at 449°C. are summarized in a plot of log $[(P_\infty - P_0)/(P_\infty - P_t)]$ vs. time, where P_0, P_t, and P_∞ represent the pressures for times $= 0$, t, and ∞, respectively. Since a straight line is obtained over essentially the entire reaction, the decomposition shows the behavior of a first-order reac-

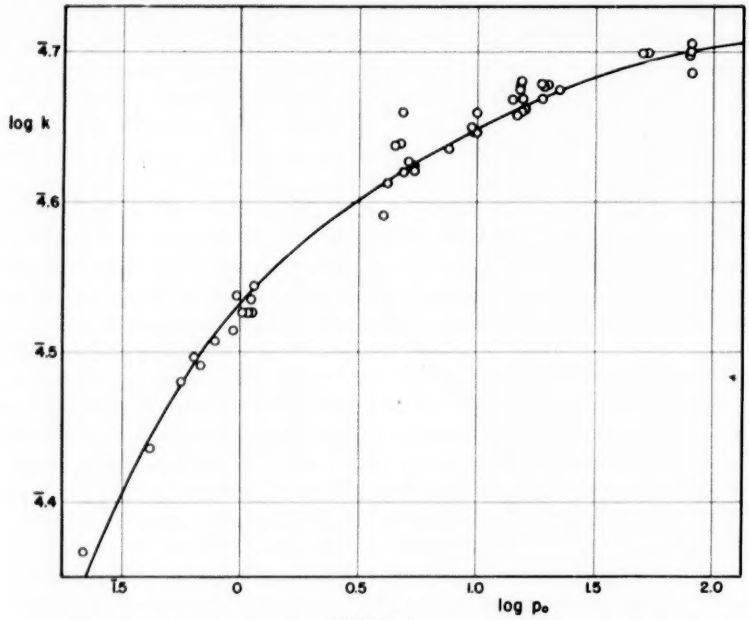

FIGURE 2

Dependence of the first order rate constant (in sec.$^{-1}$) upon the initial pressure (in mm.) at 449°C.

tion. Values for the specific reaction rate constant k obtained from the slopes of lines in plots similar to figure 1 are in good agreement with the rate constants calculated from the quarter-times ($t_{1/4}$) by means of the integrated first-order rate law. The quarter-time which has been taken as the time for 25% pressure increase (with the appropriate correction for dead space) has been used for the calculation of most of the rate constants in this work. Figure 2 shows the manner in which the values of the first-order constant change with decreasing initial pressure. At 20 mm. a decrease in the rate constant is apparent; at 0.2 mm. the rate constant is about half its value at 80 mm.

In order to secure additional information concerning the possible occurrence of free radical chain processes in the decomposition of cyclobutane at low pressures, experiments were performed in the presence of nitric oxide, propylene, or toluene.[7] As the typical results in table 2 show, there is no noticeable inhibition with any of these substances. The slightly lower values of the quarter-time, indicating higher rate constants, in the presence of propylene can be explained at least in part as the effect of an inert gas tending to restore the rate constant to the high pressure value.

TABLE 1

COMPARISON OF THE OBSERVED AND CALCULATED COMPOSITIONS OF THE REACTION MIXTURE

TEMP., °C.	INITIAL PRESSURE, MM.	CC. OF GAS AT N. T. P. (AT APPROXIMATELY HALF DECOMPOSITION) ETHYLENE OBS.	ETHYLENE CALC.	CYCLOBUTANE OBS.	CYCLOBUTANE CALC.
449.0	15.17	4.03[a]	4.03	1.75[a]	1.78
449.0 *p*	15.73 *p*	3.60[a]	3.61	1.61[a]	1.63
		VOLATILE AT −137°C.		VOLATILE AT 25°C.	
458.9	15.04	3.82	3.85	...	...
419.8	15.02	3.92[b]	3.94	1.93	1.97

[a] Obtained from the measured volumes of gas and the mass spectrometric analyses.
[b] At least 99.1% unsaturated hydrocarbon according to the absorption in activated sulfuric acid.
p = packed reaction vessel.

TABLE 2

DECOMPOSITION OF CYCLOBUTANE IN THE PRESENCE OF NITRIC OXIDE, TOLUENE, OR PROPYLENE

TEMP., °C.	ADDED SUBST.	INITIAL PRESSURE, MM. ADDED SUBST.	INITIAL PRESSURE, MM. CYCLOBUTANE	$t_{1/4}$, MIN.
449.0	....	...	15.58	10.3
449.0	....	...	15.36	10.2
449.0	NO	0.04	14.90	10.2
449.0	NO	0.31	15.02	10.1
449.1	$C_6H_5CH_3$	2.06	13.32	10.0
449.0	$C_6H_5CH_3$	4.09	15.22	10.3
449.0	C_3H_6	7.50	14.98	9.7
449.0	C_3H_6	7.79	15.44	9.9

Other preliminary experiments in which sufficient pressures of ethane or ethylene were added to 5 mm. of cyclobutane to produce a total pressure of about 65 mm. indicated that the rate constant can be raised to approximately its value in the 50–80 mm. pressure range by the addition of these presumably inert gases. Nitrogen, hydrogen, and carbon dioxide had a similar, though apparently smaller, influence. The experiments with added ethylene suggest that during the course of an experiment at a low pressure, such as the one shown in figure 1, the ethylene formed from the decompos-

ing cyclobutane will act as an added inert gas and assist in maintaining the first-order rate constant as the pressure of the undecomposed cyclobutane diminishes.

In view of the fact that the decomposition of dimethyl ether can be induced by free radicals,[8] a test for free radicals in the decomposition of cyclobutane was made by allowing 0.7 and 5 mm. of cyclobutane to decompose in the presence of 15 and 10 mm., respectively, of dimethyl ether at 449°C. The pressure measurements indicated that no induced decomposition occurred; therefore, no evidence for free radicals was obtained. It might be noted also that the decomposition of a molecule of cyclobutane into two molecules of ethylene by a non-chain process appears to be particularly favorable on the basis of the principle of least motion proposed by Rice and Teller.[9]

Since all of the evidence obtained thus far indicates that the decomposition of cyclobutane is a quasi-unimolecular reaction, the decrease in the rate constant at low pressures and the effect of inert gases will be investigated further.

Summary.—The homogeneous thermal decomposition of cyclobutane has been studied in the pressure range 0.2–80 mm. at 449°C. The reaction, which gives two moles of ethylene per mole of cyclobutane without significant side reactions, is first order and is not inhibited by nitric oxide, propylene, or toluene. With decreasing initial pressure the first-order rate constant falls off; at 0.2 mm. it is about half of its value at 80 mm. On the basis of the available data it is suggested that the reaction is quasi-unimolecular.

* This work has been supported by a grant from the Celanese Corporation of America.

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AN ULTRACENTRIFUGE CELL FOR PRODUCING BOUNDARIES SYNTHETICALLY BY A LAYERING TECHNIQUE

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Introduction.—During a study[1] of the analysis of mixtures of two macromolecules in the ultracentrifuge it was necessary to measure the sedimentation constant of a slow component while it was sedimenting in the presence of a fast component. Since such measurements cannot be made in a conventional manner in the ultracentrifuge, resort was made to layering techniques in which a solution containing only the fast component was layered over a mixture of the fast component at the same concentration plus the desired amount of slow component. In this way a boundary of slow component in the presence of the fast is created artificially and measurements of the movement of this boundary are observed by conventional optical

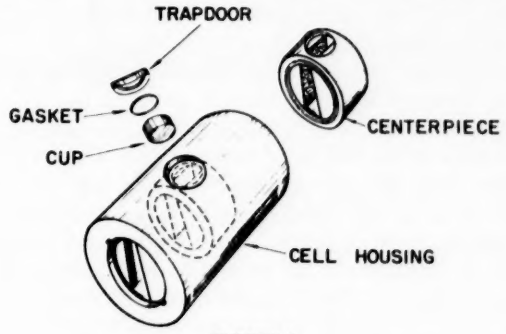

FIGURE 1
Exploded view of the synthetic boundary cell.

methods. Though boundaries can be formed in this way, the method is only qualitatively successful because manipulations of the cell and rotor subsequent to formation of the boundary cause some dissipation of the density gradient which stabilizes the boundary initially. Therefore, efforts were made to design a special cell, hereafter called a synthetic-boundary cell, in which boundaries are formed under the stabilizing influence of the centrifugal field and with the cell in its operating position in the rotor. This communication describes the design and construction of the new cell and its use in the study of mixtures in the ultracentrifuge.[2] In addition, the application of this cell to the study of the sedimentation of small molecules and to other ultracentrifuge problems is discussed.

Description of Cell.—Figure 1 is an exploded view of the synthetic-boundary cell. This cell is similar in design and size to the conventional 12 mm., 4° ultracentrifuge cell[3] manufactured by the Specialized Instruments Corporation except for changes in the centerpiece and cell housing. Instead of the small filling hole at the top of the sectoral cavity in the duralumin centerpiece, a flat bottomed hole, $^3/_8$ in. in diameter, is bored into the top of the centerpiece to a level approximately $^1/_{16}$ in. below the top of the sectoral cavity. This hole accommodates a small metal cup which has a pin-hole, $^1/_{64}$ in. in diameter, drilled in its base near the periphery. The cup is supported by the duralumin shoulders on either side of the top of the sectoral opening in the centerpiece. Concentric with the hole in the centerpiece is a larger hole through the wall of the cell housing. A curved trap door, to fit within this larger hole, is made by cutting out a section from a second cell housing and then counter boring the concave side and filing and polishing the convex surface. A rubber dam gasket of 0.007 in. thickness is placed around the cup. Compression of this gasket against the peripheral surface of the centerpiece insures a vacuum seal when the cell is pressed into the rotor hole. As an extra precaution the rotor is started turning very slowly as soon as the vacuum chamber is closed, in order to insure sealing the cell.

Use of the Cell.—In practice the cell windows, centerpiece and housing are assembled without the cup. About 0.45 cc. of the bottom (denser) solution are placed in the cell followed by a few drops (about 0.01 cc.) of mineral oil layered on the surface of the solution. A thin film of vacuum grease is rubbed onto the lower surface of the cup, which is then inserted into the centerpiece and so oriented that the pin-hole in the cup is above one of the shoulders of the centerpiece. After the lubricated cover gasket is placed against the centerpiece, the cup is filled with about 0.25 cc. of the upper (lighter) solution. The trap door, also lubricated with vacuum grease, is pressed into place as the cell is put into the rotor.

During the acceleration of the rotor the surface tension forces holding the liquid in the cup are overcome, and the liquid drains slowly through the small opening under the influence of the centrifugal field until it has all layered over the solution originally present in the bottom of the cell. In this way the boundary is created synthetically in the rotating ultracentrifuge cell.

Although sharp boundaries are obtained when the upper solution is layered directly over the lower solution, some mixing occurs as evidenced by the detection in the upper solution of solute originally in the bottom solution. The use of dyes serves as a very sensitive indicator in studying the mixing which occurs when the cup contents flow directly over the solution in the sectoral cavity. This stirring can be reduced appreciably through the use of a thin film of mineral oil which acts to break up the

stream of liquid layering over the lower solution. Mineral oil was adopted for routine use although other materials, such as finely powdered polyethylene, also serve to break up the flow of the liquid and eliminate mixing.

Depending on the amount of lubricating grease placed on the bottom of the cup and the closeness of fit between cup and centerpiece, the speed at which the layering begins (break through point) for the cup contents can be varied over a wide range. It seems desirable to grease very lightly and then seat the cup by rotating it slightly in the recessed hole of the centerpiece. In this way, with ordinary machined parts, the upper liquid begins flowing at about 2000-4000 r.p.m. The rate of flow of liquid, also, is affected by the amount of grease and fit of the cup in the centerpiece but, in addition, it can be varied by changing the acceleration rate. Reducing

FIGURE 2

Sedimentation of BSV in presence of TMV in synthetic boundary cell. Sedimentation is to the right and the leading boundary is the synthetically created BSV boundary. Pictures taken at 0, 12, 24 and 36 min. after reaching speed, respectively. Upper solution: TMV, 0.8 g./100 cc; lower solution: TMV, 0.8 g./100 cc and BSV, 0.3 g./100 cc.

the accelerating rate of the rotor after the break through point decreases the flow rate so that about 30-60 seconds are required to empty the cup completely.

Results with Mixtures.—Figure 2 shows a series of ultracentrifuge diagrams from which the sedimentation rate of a slow component, bushy stunt virus (BSV), is determined while sedimenting in the presence of a faster component, tobacco mosaic virus (TMV). Although the concentration gradient across the synthetic boundary is only 0.3 per cent, corresponding to a density difference of about 8×10^{-4} g./cc., the initial boundary is very good and the cell performed satisfactorily throughout the run. It should be noted that despite the very small density difference there is considerable initial stabilization of the boundary owing to the existence of the centrifugal field while the boundary is being formed. By use of this

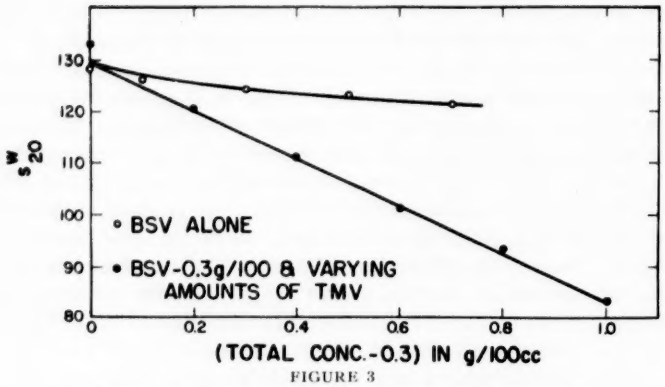

FIGURE 3

Sedimentation constant of BSV as a function of concentration.

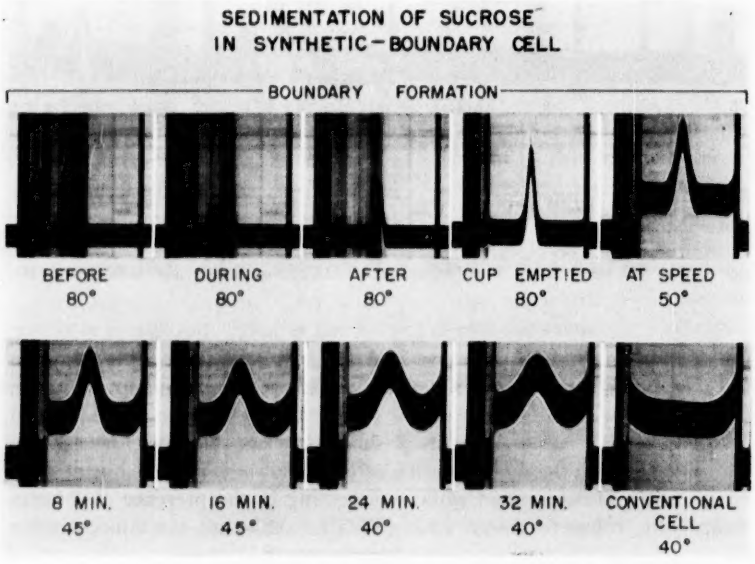

FIGURE 4

Sedimentation of sucrose in water. Upper solution: H_2O; lower solution: 1 % sucrose in H_2O. (*a*) Prior to formation of boundary; (*b*) during formation of boundary; (*c*) immediately after formation of boundary; (*d*) cup emptied; (*e–i*) 0, 8, 16, 24 and 32 min. after reaching speed, respectively; (*j*) 1% sucrose in conventional analytical cell after 32 min.; bar angles given beneath pictures.

cell a series of runs at a fixed concentration of slow component and different concentrations of fast component was made, and the hydrodynamic interactions experienced by molecules sedimenting in the presence of larger molecules was studied for the first time. In figure 3 are shown the results for BSV sedimenting by itself at different concentrations and also in the presence of different concentrations of TMV.

The Sedimentation of Small Molecules.—Direct determinations of sedimentation rates of molecules with sedimentation constants less than about 1.0S are not currently feasible by ordinary ultracentrifugation methods because of the limited resolving power of existing ultracentrifuges.[4] For small, highly diffusible molecules the diffusion from the solution toward the meniscus and from the sedimented molecules at the bottom of the cell makes it difficult to produce measurable sedimentation boundaries,

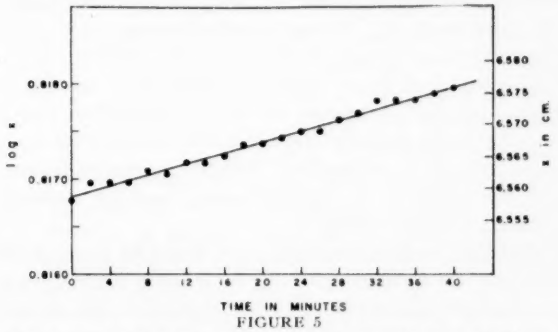

FIGURE 5

Plot of log x and x (where x is in cm. from axis of rotation) vs. t (in minutes) for sedimentation of sucrose in water.

where a boundary is defined as the transition zone between pure solvent on one side and a solution of uniform concentration (plateau region) on the other. Before sedimentation of the molecules is sufficiently prolonged to yield a clear supernatant, the plateau region is distorted by diffusion from the bottom of the cell.[5] When the synthetic-boundary cell is used, however, the period of time before distortion of the plateau region is available for the determination of sedimentation rates, since a clear supernatant is present immediately after the formation of the boundary. Measurements of the boundary position can then be made until diffusion makes the supernatant no longer free of solute or the plateau region is disturbed. Depending upon the physical properties of the solute molecules the volumes of upper and lower solutions can be varied so as to minimize the disturbances caused by diffusion.

Although the movement of the synthetic boundary by sedimentation may be extremely small, it can be measured quantitatively by reading the

photographic plates with a microcomparator. Figure 4 shows a series of ultracentrifuge patterns obtained during the formation of a boundary between water and a 1 per cent solution of sucrose. In addition, some of the pictures obtained after the rotor reached a speed of 59,780 r.p.m. are also presented. A sedimentation constant of 0.23S was determined by reading the plates by means of a microcomparator and plotting the logarithm of the distance of the boundary from the axis of rotation against time (Fig. 5). The meniscus position is usually determined for each photographic frame and is found to be constant during any one run within about ± 0.002 cm.

Discussion.—Results to date with sucrose and larger molecules are sufficiently satisfactory to encourage the feeling that the sedimentation velocity method may provide useful information about molecular weights of molecules which are too large for study by standard thermodynamic methods and too small for study by the usual methods used for macromolecules.

Preliminary experiments indicate that this new cell will also prove to be an extremely useful tool in the study of problems in ultracentrifugation such as backward flow of solvent during sedimentation of macromolecules,[6] convection in the sedimentation of mixtures,[7] analysis of equilibrium systems,[8] hydrodynamic interactions during the sedimentation of small molecules in the presence of large molecules, and the determination of differential sedimentation constants.

The cell described in this communication is one of many possible types which can be used for forming synthetic boundaries while the centrifuge is operating. Further work is in progress to improve the design so as to control more precisely the break through point and rate of flow of cell contents. Efforts are also being made to increase the effective sedimentation path in order to increase the accuracy of determining sedimentation constants of small molecules.

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THE BEADEX LOCUS IN DROSOPHILA MELANOGASTER: THE GENOTYPIC CONSTITUTION OF Bx^r

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Communicated by R. E. Clausen, September 2, 1952

It is known that the sex-linked Beadex (*Bx*) wing mutants in *Drosophila melanogaster* are of two types: dominant to and recessive to wild-type.[1] On the basis of experiments in which crossing over was demonstrated between Bx^1 (dominant) and Bx^r (recessive),[2] it was determined that Bx^1 represents a simple mutation of Bx^+, while Bx^r is a tandem duplication in which Bx^+ is duplicated. It was postulated that the wild-type alleles of the recessive mutants small eye (*sy*) and fused (*fu*) wing veins are also duplicated in Bx^r. It is the purpose of this report to confirm the postulate which includes fu^+ and sy^+ in duplicate as components of Bx^r.

The mutants *sy*, *Bx* and *fu* have been localized at positions 59.2, 59.4 and 59.5 on the X chromosome.[1] Assuming Bx^r to be a tandem duplication for all three loci, the precise Bx^r genotype would be represented by the notation: $sy^+\ Bx^+\ fu^+\ sy^+\ Bx^+\ fu^+$. To test whether the fu^+ locus is included in the Bx^r duplication the following experiments were performed using the mutants Bx^r, Bx^1, *fu* and *f* (recessive forked bristles-locus 56.7). From ♀♀ of the genotype $Bx^r/f\ Bx^1\ fu$ exceptional ♂ progeny are expected depending upon the pairing and crossing over between Bx^1 and Bx^+ of the duplication.

If Bx^1 pairs with Bx^+ in the left section of the Bx^r duplication as follows:

$$\begin{array}{l} \qquad sy^+ \ldots Bx^+ \ldots fu^+ \ldots sy^+ \ldots Bx^+ \ldots fu^+ \\ f \ldots\ldots sy^+ \ldots Bx^1 \ldots fu \end{array}$$

crossing over between Bx^1 and *fu* will produce exceptional ♂♂ of the genotypes $f\ sy^+\ Bx^1\ fu^+$ $sy^+\ Bx^+\ fu^+$ and $sy^+\ Bx^+\ fu$. Crossovers between sy^+ and Bx^1 produce no individuals phenotypically separable from crossovers between *f* and sy^+.

Conversely if Bx^1 pairs with Bx^+ in the right section of the Bx^r duplication as follows:

$$\begin{array}{l} sy^+ \ldots Bx^+ \ldots fu^+ \ldots sy^+ \ldots Bx^+ \ldots fu^+ \ldots \\ \qquad\qquad\qquad f \ldots\ldots\ldots sy^+ \ldots Bx^1 \ldots fu \ldots \end{array}$$

crossing over between sy^+ and Bx^1 will produce exceptional ♂♂ of the genotypes $sy^+\ Bx^+\ fu^+$ $sy^+\ Bx^1\ fu$ and $f\ sy^+\ Bx^+\ fu^+$. Crossovers between Bx^1 and *fu* produce no individuals phenotypically separable from crossovers occurring to the right of *fu*.

It is to be noted that if Bx^r is a duplication for fu^+ then the exceptional

♂♂ *f Bx*^{1} *Bx*$^+$ and *Bx*$^+$ *Bx*^{1} should be *fu*$^+$ in phenotype. If *fu*$^+$ is not in the duplication, the *Bx*$^+$ *Bx*^{1} ♂♂ would be *fu*. Among ca. 5000 ♂ progeny of ♀ ♀ *Bx*^{r}/ *f Bx*^{1} *fu*, 7 ♂♂ *f Bx*^{1} *Bx*$^+$ and 2 ♂♂ *Bx*$^+$ *Bx*^{1} were recovered; all were *fu*$^+$ in phenotype thereby fulfilling expectation. Moreover, if the genotypes of the *f Bx*^{1} *Bx*$^+$ and *Bx*$^+$ *Bx*^{1} ♂♂ are as noted, then it should be possible to recover the *fu* mutant from *Bx*$^+$ *Bx*^{1} but not from *f Bx*^{1} *Bx*$^+$.

Experiments in which this proposition was tested were carried out in the following manner. Females of the genotypes (1) *f Bx*^{1} *Bx*$^+$/*Bx*$^+$ *car* (recessive carnation eye color—locus 62.5) and (2) *Bx*$^+$ *Bx*^{1}/*f Bx*$^+$ *car* were obtained. Assuming the constitution of *Bx*$^+$ *Bx*^{1} and *f Bx*^{1} *Bx*$^+$ chromosomes to be as noted above, the following exceptional ♂ progeny are expected. In (1), if pairing occurs between *Bx*^{1} and *Bx*$^+$ as follows:

$$f \ldots sy^+ \ldots Bx^1 \ldots fu^+ \ldots sy^+ \ldots Bx^+ \ldots fu^+$$
$$sy^+ \ldots Bx^+ \ldots fu^+ \ldots car$$

crossing over between *Bx*^{1} and *fu*$^+$ will produce exceptional ♂ progeny *f sy*$^+$ *Bx*^{1} *fu*$^+$ *car* and *sy*$^+$ *Bx*$^+$ *fu*$^+$ *sy*$^+$ *Bx*$^+$ *fu*$^+$ (phenotypically *Bx*^{r}), with no *fu* ♂♂ being recovered. Crossovers between *sy*$^+$ and *Bx*^{1} produce no exceptional progeny. Similarly, pairing of *Bx*$^+$ with *Bx*$^+$ accompanied by crossing over within the paired section produces no exceptional progeny.

In (2) if pairing occurs between *Bx*^{1} and *Bx*$^+$ as follows:

$$sy^+ \ldots Bx^+ \ldots fu^+ \ldots sy^+ \ldots Bx^1 \ldots fu$$
$$f \ldots\ldots\ldots sy^+ \ldots Bx^+ \ldots fu^+ \ldots car$$

crossing over between *sy*$^+$ and *Bx*^{1} will produce exceptional ♂ progeny *f sy*$^+$ *Bx*^{1} *fu* and *sy*$^+$ *Bx*$^+$ *fu*$^+$ *sy*$^+$ *Bx*$^+$ *car* (phenotypically *Bx*^{r} *car*). In this cross the mutant *fu* would be recovered. Crossing over between *Bx*^{1} and *fu* will produce ♂♂ *f sy*$^+$ *Bx*$^+$ *fu* and *sy*$^+$ *Bx*$^+$ *fu*$^+$ *sy*$^+$ *Bx*^{1} *fu*$^+$ *car*. (The latter is inseparable phenotypically from crossovers between *fu*$^+$ and *car* and therefore not detectable.) Pairing of *Bx*$^+$ with *Bx*$^+$ accompanied by crossing over within the paired section produces no exceptional progeny.

The results obtained from crosses (1) and (2) only partly bear out expectation. In cross (1) among 6210 F_1 ♂♂, exceptional ♂♂ as follows were obtained: 4 *Bx*^{r}, 4 *f Bx*^{1} *fu car*, 1 *f Bx*^{1} *car*. In cross (2) among 6422 F_1 ♂♂, exceptional ♂♂ as follows were obtained: 2 *Bx*^{r} *car*; 3 *f Bx*^{1} *fu*; 2 *f fu*. These results substantiate the hypothesis that *fu*$^+$ is duplicated in *Bx*^{r}. However, the recovery of *fu* ♂♂ in cross (1) does not agree with expectation and merits further consideration here. These results suggest that not only is *fu*$^+$ duplicated in *Bx*^{r}, but at least one additional gene locus to the right of *fu*$^+$ is similarly duplicated. Since no mutants between *fu* and *car* have been reported,[1] this gene locus will be

designated as *x*. It is now possible to reconstruct the $Bx^r/f\ Bx^1\ fu$ cross by describing Bx^r as a tandem duplication with the notation $sy^+\ Bx^+\ fu^+\ x\ sy^+\ Bx^+\ fu^+\ x$. In the cross of ♀ ♀ $Bx^r/f\ Bx^1\ fu$ only the following pairing scheme need be reconsidered. (Pairing of Bx^1 with Bx^+ in the right section of Bx^r will produce results identical to those considered previously.)

$$\begin{array}{l} sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \ldots sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \\ f \ldots\ldots sy^+ \ldots Bx^1 \ \ldots fu \ \ldots\ldots x \end{array}$$

Crossovers between Bx^1 and *fu* will produce exceptional ♂♂ of the genotypes $sy^+\ Bx^+\ fu\ x$ and $f\ sy^+\ Bx^1\ fu^+\ x\ sy^+\ Bx^+\ fu^+\ x$ while crossovers between *fu* and *x* will produce ♂♂ $sy^+\ Bx^+\ fu^+\ x$ and $f\ sy^+\ Bx^1\ fu\ x\ sy^+\ Bx^+\ fu^+\ x$. Thus the assumption of *x* duplicated in Bx^r permits the extraction of two types of $f\ Bx^1\ Bx^+$ ♂♂; those carrying *fu* and fu^+ and those with two fu^+ loci. Phenotypically the two types are inseparable. It is obvious that the ♀ ♀ of cross (1) were derived from a ♂ of the former type. If the pairing scheme of cross (1) is considered in this light as follows:

$$\begin{array}{l} f \ldots\ldots sy^+ \ldots Bx^1 \ \ldots fu \ \ldots\ldots x \ldots sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \\ \qquad sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \ldots\ldots car \end{array}$$

crossovers between Bx^1 and *fu* will result in ♂♂ $f\ sy^+\ Bx^1\ fu^+\ x\ car$ (phenotypically $f\ Bx^1\ car$) and $sy^+\ Bx^+\ fu\ x\ sy^+\ Bx^+\ fu^+\ x$ (phenotypically Bx^r) while crossovers between *fu* and *x* will result in ♂♂ $f\ sy^+\ Bx^1\ fu\ x\ car$ (phenotypically $f\ Bx^1\ fu\ car$) and $sy^+\ Bx^+\ fu^+\ x\ sy^+\ Bx^+\ fu^+\ x$ (phenotypically Bx^r). This formulation fully accounts for the results obtained in cross (1), as well as those of cross (2) and it can be concluded that Bx^r is duplicated for Bx^+, fu^+ and *x*.

That the sy^+ locus is duplicated in Bx^r was demonstrated in much the same fashion. From ♀ ♀ of the genotype $Bx^r/sy\ Bx^1\ car$ exceptional ♂ progeny are again expected depending upon pairing and crossing over between Bx^1 and Bx^+ of the duplication.

If Bx^1 pairs with Bx^+ in the left section of the duplication as follows:

$$\begin{array}{l} sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \ldots sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \\ sy \ \ \ldots Bx^1 \ \ldots fu^+ \ldots x \ \ldots\ldots car \end{array}$$

crossing over in the interval Bx^1-*x* will produce exceptional ♂♂ of the genotypes $sy^+\ Bx^+\ fu^+\ x\ car$ and $sy\ Bx^1\ fu^+\ x\ sy^+\ Bx^+\ fu^+\ x$. Crossovers between *sy* and Bx^1 produce no individuals phenotypically separable from crossovers occurring to the left of *sy*.

Conversely if Bx^1 pairs with Bx^+ in the right section of the duplication as follows:

$$sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \ldots \quad sy^+ \ldots Bx^+ \ldots fu^+ \ldots x$$
$$sy \ldots Bx^1 \ldots fu^+ \ldots car$$

crossing over between *sy* and Bx^1 will produce exceptional ♂♂ of the genotypes *sy* Bx^1 fu^+ *x* and sy^+ Bx^+ fu^+ *x* sy^+ Bx^1 fu^+ *x car*. Crossovers in the interval Bx^1-*x* produce no individuals phenotypically separable from crossovers between *x* and *car*.

It is to be noted that if Bx^r is duplicated for sy^+ then the exceptional ♂♂ Bx^+ Bx^1 *car* and Bx^1 Bx^+ should be sy^+ in phenotype. If sy^+ is not in the duplication, Bx^1 Bx^+ ♂♂ would be *sy*. Among ca. 3000 ♂ progeny of ♀ ♀ Bx^r/*sy* Bx^1 *car*, 3 ♂♂ Bx^1 Bx^+ *car* and 2 ♂♂ Bx^+ Bx^1 were recovered; all were sy^+. It follows that if the genotypes of the Bx^+ Bx^1 *car* and Bx^1 Bx^+ ♂♂ are as noted, then it should be possible to recover the *sy* mutant from Bx^1 Bx^+ but not from Bx^+ Bx^1 *car*.

Experiments to test this conclusion were carried out in the following fashion. Females of the genotypes (3) Bx^1 Bx^+/*f* Bx^+ *car* and (4) Bx^+ Bx^1 *car*/*f* Bx^+ were obtained. Assuming the constitution of the Bx^1 Bx^+ and Bx^+ Bx^1 *car* chromosomes to be as noted previously, the following exceptional ♂ progeny are expected. In (3) if pairing occurs between Bx^1 and Bx^+ as follows:

$$sy \ldots Bx^1 \ldots fu^+ \ldots x \ldots sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \ldots$$
$$f \ldots sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \ldots\ldots car$$

crossing over between *sy* and Bx^1 will produce ♂♂ *sy* Bx^+ fu^+ *x car* and *f* sy^+ Bx^1 fu^+ *x* sy^+ Bx^+ fu^+ *x* (the latter indistinguishable from crossovers between *f* and sy^+). Crossing over in the interval Bx^1-*x* will produce ♂♂ *sy* Bx^1 fu^+ *x car* and ♂♂ *f* sy^+ Bx^+ fu^+ *x* sy^+ Bx^+ fu^+ *x* (phenotypically Bx^r). As noted previously pairing between Bx^+ and Bx^+ accompanied by crossing over results in no exceptional progeny.

In (4) if pairing occurs between Bx^1 and Bx^+ as follows:

$$sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \ldots sy^+ \ldots Bx^1 \ldots fu^+ \ldots x \ldots car$$
$$f \ldots\ldots sy^+ \ldots Bx^+ \ldots fu^+ \ldots x \ldots$$

crossing over between sy^+ and Bx^1 will result in exceptional ♂♂ of the genotypes *f* sy^+ Bx^1 fu^+ *x car* and sy^+ Bx^+ fu^+ *x* sy^+ Bx^+ fu^+ *x* (phenotypically Bx^r). Crossovers in the Bx^1-*x* interval will result in ♂♂ inseparable from crossovers between *x* and *car*. Again pairing between Bx^+ and Bx^+ accompanied by crossing over results in no exceptional progeny.

The results obtained from crosses (3) and (4) were those expected from the hypothesis. From cross (3) among 4419 F_1 ♂♂, 8 ♂♂ *sy car*, 7 ♂♂ *sy* Bx^1 *car* and 2 ♂♂ *f* Bx^r were recovered while from cross (4) among 5702 F_1 ♂♂, 3 ♂♂ *f* Bx^1 *car* were recovered. These results demonstrate that sy^+ is duplicated in Bx^r.

In summary it can be stated that in Bx^r the sy^+, Bx^+, fu^+ loci as well as x, a locus to the right of fu^+, are duplicated.

[1] Bridges, C. B., and Brehme, K. S., "The Mutants of *Drosophila melanogaster*," *Carnegie Inst. Wash. Publ.* **552,** (1944).

[2] Green, M. M., *Genetics* (in press) (1952).

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*THE PSEUDOALLELISM OF WHITE AND APRICOT IN DROSOPHILA MELANOGASTER**

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The classical example of multiple allelism is the series of eye-color mutants at the white (w) locus in *Drosophila melanogaster*. The alternative interpretation of this series, namely, that it is made up of "pseudoalleles," or closely linked genes with similar effects, has usually been considered ruled out by two kinds of evidence. In the first place early attempts to resolve the series by crossing over failed in spite of numerous tests involving most of the mutants available at the time.[1-4] Secondly, a heterozygote for two different mutant genes of the series does not have the pheno type expected for non-allelic genes, namely, wild-type (or red) eye color, but instead has a mutant eye color which is usually intermediate between the colors of the two respective homozygotes. In recent years, however, several cases have been found in which non-allelic genes give a positive phenotypic test for allelism by virtue of a position effect.[5-7] In such cases, which have been termed "position pseudoalleles,"[7] mutant genes at the different loci (say, a and b) give a mutant phenotype in the $a\ +/+\ b$ heterozygote, but a wild-type, or more nearly wild-type, phenotype in the $a\ b/+\ +$ heterozygote.

With the above considerations in mind and with the aid of more adequate techniques for studying crossing over than were available in the early studies, the white gene and its so-called "allele," apricot, have been reinvestigated. This paper presents the evidence that these two genes occupy separate loci and that they constitute another example of position pseudoallelism. In what follows, the apricot gene, formerly symbolized as w^a will be designated by a new symbol, namely, *apr*.

In order to investigate the possibility of crossing over between w and *apr*, females with attached-X chromosomes were employed so that the two complementary products from any such crossing over would sometimes be recoverable simultaneously in a single individual. The first step was the

construction of attached-X's heterozygous for *w*, *apr* and suitable marker genes on either side of the *w* locus (at 1.5 in the X-chromosome). The markers chosen were yellow (*y*, at 0.0; body and bristle color); yellow-2 (y^2, allelic and dominant to *y*; with black instead of yellow bristles); scute (*sc*, at 0.0+; missing bristles); split (*spl*, at 3.0±; abnormal bristles and rough eyes); and echinus (*ec*, at 5.5; enlarged facets).[8,9] By the standard method[10] of using attached-X triploid females, an attached-X diploid female of the following or "Type-A" constitution was obtained (see Fig. 1): y^2 *apr ec*/*y sc w spl*. Such a female is phenotypically yellow-2 and "dilute-apricot"; the latter is a pinkish yellow eye color which is often distinguishable from the "apricot" or yellowish pink color of homozygous *apr*. A Type-A female produces predominantly three phenotypic

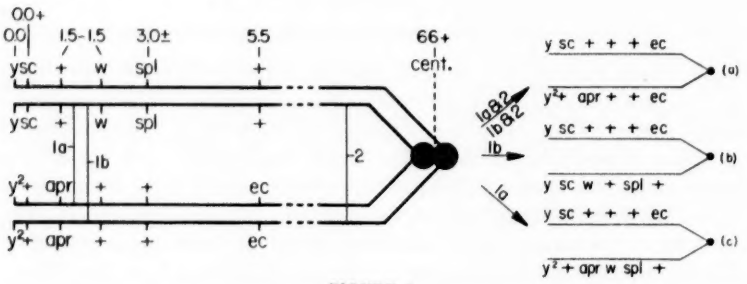

FIGURE 1

The diagram on the left is the attached-*X* constitution of Type A from which the majority of crossovers between *apr* and *w* were detected. The types of exchange which give rise to such crossovers are labeled: 1*a* (reciprocal) and 1*b* (non-reciprocal). On the right are shown the resultant three classes of crossover-containing attached-*X*'s. An additional exchange (exchange 2) between the locus of *ec* and the centromere (cent.) is required for the production of class (*a*).

classes of attached-X progeny; namely, (1) yellow-2 dilute-apricot, (2) yellow-2 apricot echinus and (3) yellow scute white split. Females belonging to Class 1 prove to be primarily of Type-A constitution like the mother. Classes 2 and 3 are the diagnostic ones for a Type-A constitution since they correspond to the two kinds of homozygotes with respect to the original linked sets of mutant genes in the mother, namely, y^2 *apr ec* and *y sc w spl*, respectively. Such homozygous classes result from certain types of exchange(s) in the region between the *ec* locus and the centromere (see e.g., exchange numbered 2 in Fig. 1). Since the latter region is over 50 map units in length, a single Type-A attached-X female usually produces numerous daughters belonging to each of the above homozygous classes; conversely, the constitution of an individual female with respect to the above group of genes is usually readily determined from inspection of the

two principal classes of homozygous daughters. Although certain types of exchange within the $y - ec$ region lead to constitutions and phenotypic classes other than those discussed above, the relatively short genetic length of this region makes such exceptions to the above rules either infrequent or absent in the progeny of a single female.

Type-A attached-X females were next made heterozygous for chromosomal rearrangements in the second and third chromosomes, since such an autosomal constitution is known[11] to be very effective in causing an increase in crossing over in the X-chromosome. For the second chromosome the two Curly (*Cy*) inversions[9] were chosen (marked by the dominant wing-mutant, *Cy*).[12] For the third chromosome a new complex rearrangement of x-ray origin was chosen. This rearrangement involves five breakage points[13] distributed throughout the third chromosome and is inseparably associated with a dominant bithorax-like change, termed Ultrabithorax-130 (Ubx^{130}). The Ubx^{130} heterozygote is readily classified on the basis of an enlarged distal segment of the haltere (while the homozygote is lethal).

The first indication of crossing over between *w* and *apr* came in the offspring of a preliminary mating, Mating 1, in which the parental yellow-2 dilute-apricot females were heterozygous for only the Ubx^{130} rearrangement. All of such females were descendants of the original Type-A female described above. The parental males in Mating 1 were heterozygous for a *Cy* chromosome which was known to carry both of the *Cy* inversions. In 19 cultures of this mating the individually mated parental females proved to have had a Type-A constitution. One of these cultures produced a single yellow-2 red echinus female. This female in turn produced 23 daughters, of which 12 were phenotypically like the mother, while 6 were yellow scute red echinus and 5 were yellow-2 apricot echinus. This is the result to be expected if the original red-eyed fly had the following constitution: $y\ sc\ ec/y^2\ apr\ ec$. As the result of detachment of the attached-X's in one of the above six homozygous *y sc ec* females, *y sc ec* males were obtained and these also proved to have red eyes. The *y sc ec* chromosome was then tested against a known deficiency for the white gene (Notch-8)[9] and the resultant females also had the wild-type eye color. Thus, the *y sc ec* chromosome appears to be completely wild-type with respect to the white region. The simplest interpretation of the origin of this chromosome is that it represents a wild-type crossover chromatid which resulted from an exchange between the *w* locus and an *apr* locus lying to the left of *w* (see exchange 1a or 1b of Fig 1). The presence of the $y^2\ apr\ ec$ chromosome in the original red-eyed female is readily accounted for by assuming a non-reciprocal exchange (see exchange 2 of Fig. 1) between the *ec* locus and the centromere.

A search for additional red-eyed flies was continued among the progeny

of the second and final type of mating, Mating 2, in which the parental yellow-2 dilute-apricot females were heterozygous for *Cy* as well as Ubx^{130}. There were four groups of such parental females. The first group was selected from among the daughters of the 19 Type-A cultures of Mating 1, described above; the second group was selected from among the daughters of only those females belonging to the first group which proved to be of Type-A constitution; while the third and fourth groups were similarly selected from the daughters of the second and third groups, respectively. The parental females of Mating 2 were in every case individually mated to males carrying the sex-linked mutant genes, y^{3ld}, sc^8, *apr*, *B* (Bar) and lz^s (lozenge-spectacled).[9] The *B* mutant served here to identify any free-X daughters, which result rarely from detachment of the attached-X's in the mother. The nearly white (*apr* lz^s) eye color of this type of male facilitated the search for red-eyed flies among the progeny as a whole. (The yellow body color of this type of male served a similar purpose with respect to any non-yellow progeny, which might result from crossing over between *y* and y^2, but none of these was found.)

TABLE 1

THE NUMBER AND CLASSES OF RED-EYED DAUGHTERS OF y^2 *apr ec*/*y sc w spl*, OR "TYPE-A," ATTACHED-X FEMALES (HETEROZYGOUS FOR AUTOSOMAL REARRANGEMENTS) FROM MATING 2. THE CLASSES CORRESPOND AS LETTERED TO THOSE SHOWN IN FIGURE 1

CLASS	PHENOTYPE	CORRESPONDING GENOTYPE	NUMBER OF FLIES
(a)	yellow-2 echinus	$\frac{y\ sc\ +\ +\ +\ ec}{y^2\ +\ apr\ +\ +\ ec}$	2
(b)	yellow scute	$\frac{y\ sc\ +\ +\ +\ ec}{y\ sc\ +\ w\ spl\ +}$	4
(c)	yellow-2	$\frac{y\ sc\ +\ +\ +\ ec}{y^2\ +\ apr\ w\ spl\ +}$	4

Among 897 fertile cultures[14] of Mating 2, a total of 794 cultures proved to have had parental females which were either of Type-A constitution or were heterozygous for at least *y*, *sc* and *spl*, as well as *apr* and *w;* while the remaining 103 cultures had to be discarded either because of too few progeny or because of an insufficiently marked maternal constitution. From the above 794 adequately constituted cultures, there was a total of 12 independent occurrences of red-eyed females among an estimated 40,100 attached-X offspring. Ten of these 12 red-eyed females came from Type-A mothers. As shown in table 1, these ten were distributed among three phenotypic classes with respect to the sex-linked marker genes. Each of the two yellow-2 red echinus flies of Class (a) proved on progeny testing to have had attached-X's made up of a *y sc ec* or wild-type crossover chromosome and a y^2 *apr ec* chromosome, as in the first case of a red-eyed fly from Mating 1, above. Of the four yellow scute red females of Class (b), one died and each of the others proved to have had a *y sc ec* chromosome

associated with a *y sc w spl* chromosome. Class (b) individuals are those expected following a single non-reciprocal exchange (see exchange Ib of Fig. 1) between *apr* and *w*. The four yellow-2 females of Class (c) each produced the following principal classes of homozygous attached-X daughters: yellow scute red echinus and yellow-2 white split. Each of the four females of Class (c) must therefore have represented a case of simultaneous recovery of a *y sc ec* wild-type crossover chromosome and the complementary double mutant, or y^2 *apr w spl*, crossover chromosome. Of the two remaining red-eyed flies among the above group of 12, one was yellow-2 in phenotype and arose from a y^2 *apr*/*y sc w spl* mother (identical in constitution with Type A except for the loss of *ec*). The principal two classes of homozygous daughters of this red-eyed female were phenotypically yellow scute red and yellow-2 white split. The other red-eyed fly, also of yellow-2 phenotype, came from a y^2 *w spl*/*y sc apr ec* mother (identical with Type A except for an interchanging of the markers distal to *apr*). The principal two classes of homozygous daughters of this latter red-eyed female were yellow-2 red echinus and yellow scute white split in phenotype. Thus, each of the above two red-eyed flies must have represented a case in which both crossover chromatids from an exchange between *apr* and *w* had been recovered simultaneously.

The above analysis of the 12 red-eyed attached-X females from Mating 2 actually provided a total of 18 crossovers between *apr* and *w;* that is, each of the 12 carried a wild-type crossover while six of them carried the complementary double mutant crossover, as well. The observed amount of crossing over was 0.03% (12/40,100 ±). This value probably overestimates the standard map distance between *apr* and *w*, since the observed total amount of crossing over in the whole region from *y* to *spl* in these experiments was calculated to be 11.5% (based on a complete phenotypic analysis of 11,985 attached-X progeny from a large sample of Type-A cultures of Mating 2), or nearly four times the standard value of 3.0±%.[15] This increase, however, was not distributed uniformly over the *y* − *spl* region. Thus, the *y* − *apr* region was increased from 1.5 to 8.9% or nearly sixfold;[16] while the *w* − *spl* region increased from 1.5± to 2.6% or about 1.7 times. By assuming that crossing over in the *apr*−*w* region was increased by a factor lying within these limits, the standard map distance between *apr* and *w* can be calculated as being within the range of 1/6 to 1/1.7 of the observed value of 0.03 unit, or roughly 0.005 to 0.02.

Indirect proof was obtained above for the occurrence of six double mutant, or *apr w*, crossovers from Mating 2. In none of these cases, however, has it been possible to distinguish, phenotypically, *apr w* from *w*. Thus, the heterozygote, *apr w*/+ +, invariably has red eyes in striking contrast to the pinkish yellow eye color of *apr* +/+ *w* heterozygotes; while the *apr w* homozygote is like that of *w* in having white eyes. In

addition, two *apr w* detachment males were obtained from one of the females belonging to Class (c) of table 1. Such males also have white eyes. Similarly, *apr w/+ w* females have white eyes, and *apr w/apr +* females have pinkish yellow eyes like those of *+w/apr +*.

An attempt was next made to obtain direct proof for the presence of *apr* in the double mutant combination by searching among the progeny of *apr w/+ +* females for a dilute-apricot phenotype. For this purpose a third type of mating, Mating 3, was employed. The parental females in this case were phenotypically yellow-2 red (and were in some cases heterozygous for *Cy* and/or Ubx^{130}). Each was derived as either a second or third generation daughter of one of the original four yellow-2 red females from Mating 2 (the same one, in fact, from which the above *apr w* detachments were ultimately derived). The parental females of Mating 3 were individually mated to males which carried wild-type X-chromosomes (and were heterozygous for Ubx^{130}). To forestall, in so far as possible, any contamination of this mating with dilute-apricot attached-X females, all cultures containing such females had been purposely destroyed before Mating 3 was initiated, except for one Type-A culture in which, however, all of the flies were homozygous for the Gowen gene (*c3G*).[9] Since this gene results in complete suppression of crossing over in the female, Type-A females from this latter type of culture are readily recognizable because they breed true for the Type-A constitution and are nearly sterile. From one culture of Mating 3 in which the parental female proved to have had the constitution, *y sc ec/y^2 apr w spl*, a single dilute-apricot (yellow-2) female (which also happened to carry *Cy* and Ubx^{130}) arose among the otherwise red- or white-eyed progeny. The principal classes of homozygous daughters of this latter female were yellow-2 apricot echinus and yellow scute white split in phenotype; thus, the maternal constitution in this case must have been *y^2 apr ec/y sc w spl*, or Type A. In this case there must have been simultaneous recovery of the *w+* and *apr +* crossover chromatids that are to be expected following an exchange between the *apr* and *w* loci in the parental *apr w/+ +* female.

Discussion.—The above crossing-over studies have shown that at least two loci are at the basis of the so-called "multiple allelic" series of white mutants. Only brief consideration can be given here to the relation of this finding to results obtained from several other types of studies that have already been made of this series. Firstly, phenotypic studies have shown that this series is comprised of two qualitatively distinct groups of mutants. Thus, the eye color of mutants belonging to the so-called "apricot" group is darker in the male than in the female; while the converse is true for mutants of the "eosin" group. Similarly, Bridges' specific modifier, Pale, acts to darken the eye color of mutants of the apricot group, and to lighten it for those of the eosin group.[17] Moreover, it has been

pointed out[17] that "*w*" (presumably the same *w* as used in the present studies) belongs to this latter group since the associated "white" phenotype is made even lighter in color in the presence of Pale.

The above difference in properties between the apricot and eosin group of mutants will more than likely turn out to reflect a functional difference between the apr^+ and w^+ genes. Superimposed on such a difference, however, would be the position effect associated with these genes; namely, the striking phenotypic difference that exists between *apr*+/+ *w* (with pinkish yellow eye color) and *apr w*/+ + (with red eye color). This result suggests that a mutant gene at one of the loci blocks, or impairs the functioning of, the normal allele of the gene at the other locus, when both are present in the same chromosome, as in the *apr* +/+ *w* heterozygote; while no impairment of the functioning of the two different wild-type alleles is phenotypically detectable when both of these are present in the same chromosome. The simplest assumption is that the effect is one-way; that is, that the mutant gene *apr* impairs the functioning of w^+, or that *w* impairs that of apr^+. This leads to a simple model in which one of the genes controls a step A→B, and the other a step, B → C, in a biochemical reaction chain of the type: A → B → C. The position effect can then be assumed to result from a failure of substance B to diffuse readily from one chromosome to the other so that the chain of reactions in one of the chromosomes of the heterozygote is carried out more or less independently of the chain in the homologous chromosome. Further details of the application of this type of model to position pseudoallelism have been given elsewhere.[7]

Cytological studies of Panshin[18] and of others[9] have shown that the w^+ gene is located within the confines of the 3C2-3 doublet, or two-banded, structure of the salivary gland chromosomes (see Bridges' revised map[19]). Moreover, the evidence of Panshin was based on the critical method of synthesizing a deficiency for the gene in question by combining parts of two appropriate rearrangements, neither of which acts like a deficiency for that gene. In this case, the rearrangements were the white-mottled-5 (w^{m5}) and roughest-3 (rst^3) inversions in the X-chromosome. The right break-points of these two inversions are essentially similar and in the heterochromatic region. The left break-point of w^{m5} lies between 3C1 and 3C2 (Sutton[9]);[20] while that of rst^3 lies between 3C3 and 3C4 (Emmens and others[9]). Panshin combined, as the result of crossing over, the left end of w^{m5} with the right end of rst^3. The resultant chromosome is a deficiency for the 3C2-3 doublet and it proved to act like a deficiency for the w^+ gene; on the other hand, other synthesized deficiencies either for the 3C1 band, or for bands to the right of the 3C2-3 doublet, did not act as though they were deficiencies for this gene. From these results, it is probable that apr^+ also lies within the 3C2-3 doublet.

The doublet structure has been implicated before in certain cases of position pseudoallelism. As has been discussed more fully in connection with these other cases,[5,7] the probable significance of this cytological finding is that the pseudoallelic genes associated with a doublet represent an established duplication of a single ancestral gene. To explain the finding that the two members of the duplication now appear to differ in function, it has been presumed that one of the genes has, by mutation, diverged in function from the other, and, in such a way, that it now controls a reaction successive to (or, on Horowitz' hypothesis for the evolution of biochemical synthesis,[21] antecedent to) that of the original gene.

Summary.—The white (*w*) and apricot (*apr*, formerly w^a) genes of the so-called "multiple allelic" white-series in *Drosophila melanogaster* are found to be pseudoallelic genes whose order in the chromosome is *apr*−*w*. This result was based on a total of 21 crossovers between the *apr* and *w* loci. The total percentage of crossing over between them was 0.03%, under conditions giving two to six times the normal amount of crossing over in the surrounding regions. The standard map distance between *apr* and *w* is inferred to be in the range of 0.005 to 0.02 unit. The *apr* and *w* genes are a typical example of position pseudoallelism. Thus, a position effect is present, whereby *apr* +/+ *w* females have a pinkish yellow eye color, while *apr w*/+ + females have a wild-type or red eye color. The use of attached-X chromosomes enabled the *apr w*, or double mutant, combination to be derived simultaneously with the complementary wild-type crossover in a total of six instances. The results are discussed in relation to phenotypic and cytological studies of the white-series, and also in relation to other studies of position pseudoallelism.

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[14] The author is indebted for technical assistance in the preparation of these matings to W. Gencarella.

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ON THE NUMBER OF BOUND STATES IN A CENTRAL FIELD OF FORCE

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1. The present note contains some fairly elementary remarks concerning the number of bound state solutions of the Schrödinger equation

$$\nabla^2\psi + E\psi = V(r)\psi$$

for a central field of force, more specifically, the number n_l of bound state solutions of the radial wave equation

$$\phi'' - l(l+1)r^{-2}\phi + E\phi = V(r)\phi \tag{1}$$

for angular momentum l. We assume the integral

$$I = \int_0^\infty r\,|V(r)|\,dr \tag{2}$$

to be *finite*, and we wish to estimate n_l in terms of I. (In the units chosen V has the dimension (length)$^{-2}$, so that I is dimensionless.) R. Jost and A. Pais (ref. 1, p. 844) have shown that no bound states occur if $I < 1$. Our aim is to derive the more general inequality

$$(2l+1)n_l < I \tag{3}$$

(equality excluded). The number n_l counts the distinct stationary energy values corresponding to equation (1). If the $(2l+1)$-fold degeneracy of each of them is taken into account it is seen that for a given angular momentum l there are less than I bound states, and no bound states occur if $l \geq \frac{1}{2}(I-1)$. The estimate (3) is best possible in the sense that for a given l potentials may be constructed which have a prescribed number n_l of bound states for that angular momentum and for which I approaches

$(2l + 1)n_l$ arbitrarily closely (see section 5 below). The whole question is taken up because the finiteness of I plays a significant role in several recent investigations on scattering theory.[1-4] (V may have any singularities consistent with a finite value of I.)

2. As is well known, n_l is the number of zeros (not counting $r = 0$) of that solution $\phi(r)$ of the equation

$$\phi'' - l(l + 1)r^{-2}\phi = V(r)\phi \tag{4}$$

($E = 0$) which vanishes at the origin. Special care must be taken with a possible bound state $E = 0$. Since I is finite any solution of (4) has the following asymptotic behavior at infinity. The expression $r^{-(l+1)}\phi(r)$ always approaches a finite limit, say λ, as $r \to \infty$. If $\lambda \neq 0$, $\phi(r)$ increases indefinitely. If $\lambda = 0$ the expression $r^l\phi(r)$ approaches a finite limit μ, and $\mu \neq 0$. In the latter case $\phi(r)$ is square integrable if $l > 0$, and accordingly $E = 0$ is a bound state.[5] For $l = 0$, $E = 0$ is never a bound state if I is finite. For the purpose of our discussion, however, we shall count $r = \infty$ as a zero of $\phi(r)$—even if $l = 0$—whenever $\lim_{r\to\infty} r^{-(l+1)}\phi(r) = 0$, and interpret the inequality (3) accordingly.[6]

Replace in equation (1) $V(r)$ by a potential $V_1(r)$ such that $V_1(r) \leq V(r)$ for all r, and denote by n_l' the number of bound states for the new potential. Then $n_l' \geq n_l$. We shall choose $V_1(r) = -W(r)$, where $W(r) = |V(r)|$, and study the equation

$$\phi'' - l(l + 1)r^{-2}\phi = -W(r)\phi \qquad W(r) = |V(r)|. \tag{5}$$

Denote by $\nu_1, \nu_2, \ldots, \nu_n$ ($n = n_l'$) the successive zeros of $\phi(r)$ ($0 < \nu_1 < \nu_2 < \ldots < \nu_n$), and set $\nu_0 = 0$. We shall prove

$$\int_\alpha^\beta rW(r)\,dr > 2l + 1;\ \alpha = \nu_{k-1}, \beta = \nu_k, k \geq 1. \tag{6}$$

The inequality (3) is obtained by adding the n inequalities (6), for we find then

$$I = \int_0^\infty rW(r)\,dr \geq \int_0^{\nu_n} rW(r)\,dr > n_l'(2l + 1) \geq n_l(2l + 1).$$

3. *Preliminary Remarks on $\phi(r)$.*—The solution of equation (5) which vanishes at the origin is uniquely determined up to a constant factor. As $r \to 0$, $r^{-(l+1)}\phi(r)$ approaches a finite non-vanishing limit κ. Choosing $\kappa = 1$, we find from (5)

$$\phi(r) = r^{l+1} - \int_0^r G(r, \rho)\phi(\rho)W(\rho)\,d\rho. \tag{7}$$

$\phi(r)$ is then real. Here $G(r, \rho)$ is the fundamental solution of the equation $f'' - l(l + 1)r^{-2}f = 0$, i.e., $\partial^2G(r, \rho)/\partial r^2 - l(l + 1)r^{-2}G(r, \rho) = 0$, $G(r, r) = 0$, and $\partial G(r, \rho)/\partial r = 1$ for $r = \rho$. We have

$$G(r, \rho) = (2l + 1)^{-1}H(r, \rho); \quad H(r, \rho) = r(r/\rho)^l - \rho(\rho/r)^l \quad (8)$$

if $r > 0$, $\rho > 0$. Clearly $H(r, \rho) > 0$ if $r > \rho$. Since, for $r \to 0$, $r^{-(l+1)}\phi(r) \to 1$, the integral in (7) is absolutely convergent.

In the sequel we shall need the inequality

$$H(\beta, \rho)H(\rho, \alpha) \leq \rho(H(\beta, \alpha) - V(\beta, \alpha))$$

$$V(\beta, \alpha) = 2(\alpha\beta)^{1/2}[1 - (\alpha/\beta)^{l+1/2}] > 0 \qquad (\beta > \alpha). \qquad (9)$$

To derive it consider $Z(\rho, \beta, \alpha) = \rho H(\beta, \alpha) - H(\beta, \rho)H(\rho, \alpha)$. By straight forward computation

$$\begin{aligned} Z(\rho, \beta, \alpha) &= \rho(\alpha\beta)^{1/2}\{(\rho^2/\alpha\beta)^{l+1/2} + (\alpha\beta/\rho^2)^{l+1/2} - 2(\alpha/\beta)^{l+1/2}\} \\ &= \rho(\alpha\beta)^{1/2}\{[(\rho/\sqrt{\alpha\beta})^{l+1/2} - (\sqrt{\alpha\beta}/\rho)^{l+1/2}]^2 + 2[1 - (\alpha/\beta)^{l+1/2}]\} \\ &\geq \rho V(\beta, \alpha) \end{aligned}$$

which establishes (9).

4. *Proof of* (6).—We distinguish four cases[7] according as $\alpha = 0$, $\alpha > 0$; $\beta < \infty$, $\beta = \infty$.

(*a*) $\alpha = 0$, $\beta = \nu_1 < \infty$. On the open interval $(0, \beta)$ ϕ is positive, and hence, by (7), $\phi(r) \leq r^{l+1}$. Since $\phi(\beta) = 0$ we have from (7) and (8)

$$\begin{aligned} (2l + 1)\beta^{l+1} &= \int_0^\beta H(\beta, \rho)\phi(\rho)W(\rho)d\rho \leq \int_0^\beta H(\beta, \rho)\rho^{l+1}W(\rho)\, d\rho \\ (2l + 1)\beta^{l+1} &\leq \beta^{l+1}[\int_0^\beta \rho W(\rho)\, d\rho - \int_0^\beta (\rho/\beta)^{2l+1}\rho W(\rho)\, d\rho]. \end{aligned}$$

On dividing by β^{l+1} we find (6) because the last integral is positive.

(*b*) $\alpha > 0$, $\beta < \infty$. Since $\phi(\alpha) = 0$, the derivative $\phi'(\alpha)$ does not vanish. If we replace $\phi(r)$ by $\chi(r) = \phi(r)/\phi'(\alpha)$, then $\chi(\alpha) = 0$, $\chi'(\alpha) = 1$, and hence

$$\chi(r) = G(r, \alpha) - \int_\alpha^r G(r, \rho)\chi(\rho)W(\rho)\, d\rho \qquad (10)$$

On the interval $\alpha < r < \beta$, therefore, $0 < \chi(r) \leq G(r, \alpha)$. Thus, for $r = \beta$,

$$G(\beta, \alpha) = \int_\alpha^\beta G(\beta, \rho)\chi(\rho)W(\rho)\, d\rho \leq \int_\alpha^\beta G(\beta, \rho)G(\rho, \alpha)W(\rho)\, d\rho,$$

or

$$\begin{aligned} (2l + 1)H(\beta, \alpha) &\leq \int_\alpha^\beta H(\beta, \rho)H(\rho, \alpha)W(\rho)\, d\rho \\ &\leq [H(\beta, \alpha) - V(\beta, \alpha)] \int_\alpha^\beta \rho W(\rho)\, d\rho \end{aligned}$$

[see (9)]. Division by $H(\beta, \alpha)$ establishes (6).

(*c*) $\alpha = 0$, $\beta = \nu_1 = \infty$. Here we use (7), and, as in case (*a*), $r^{l+1} \geq \phi(r) > 0$ for all positive r. By assumption, $\tau(r) = (2l + 1)r^{-(l+1)}\phi(r)$ approaches 0 as $r \to \infty$. By (7),

$$2l + 1 - \tau(r) - \int^r K(r, \rho)\phi(\rho)W(\rho)\, d\rho = 0$$

where

$$K(r, \rho) = r^{-(l+1)}H(r, \rho) = \rho^{-l}(1 - (\rho/r)^{2l+1}) < \rho^{-l}$$

Hence

$$\int_0^r \rho W(\rho)\, d\rho \geq \int_0^r \rho^{l+1}K(r, \rho)W(\rho)\, d\rho \geq 2l + 1 - \tau(r) + \int_0^r K(r, \rho)[\rho^{l+1} - \phi(\rho)]W(\rho)\, d\rho \quad (11)$$

Since the integral in (11) is non-negative and $\tau(r) \to 0$, we find at once that $\int_0^\infty \rho W(\rho)\, d\rho \geq 2l + 1$. To exclude equality we observe that there must exist two adjacent intervals $[\xi, \eta]$ and $[\eta, \zeta]$ $(\xi < \eta < \zeta < \infty)$ such that $\int_\xi^\eta \rho W(\rho)\, d\rho$ and $\int_\eta^\zeta \rho W(\rho)\, d\rho$ both exceed 1/4, say. If $r \geq \eta$, then, by (7), $r^{l+1} - \phi(r) \geq \int_\xi^\eta G(r, \rho)\phi(\rho)W(\rho)\, d\rho$ and throughout the interval $[\eta, \zeta]$ $(\rho^{l+1} - \phi(\rho))\rho^{-(l+1)} \geq c$, where c is some positive constant. For $r > \zeta$ we find from (11)

$$\int_0^r \rho W(\rho)\, d\rho \geq 2l + 1 - \tau(r) + c \int_\eta^\zeta \rho^l K(r, \rho)\rho W(\rho)\, d\rho$$

and in the limit $r \to \infty$

$$\int_0^\infty \rho W(\rho)\, d\rho \geq 2l + 1 + c \int^\zeta \rho W(\rho)\, d\rho > 2l + 1, \quad \text{q. e. d.}$$

(*d*) $\alpha > 0$, $\beta = \nu_n = \infty$. We start, as in (*b*), from equation 10, so that $G(r, \alpha) \geq \chi(r) > 0$ for $r > \alpha$. By assumption, $(2l + 1)^{-1}\theta(r) = \chi(r)/G(r, \alpha)$ approaches 0 as $r \to \infty$. From (10) we find

$$2l + 1 - \theta(r) - \int_\alpha^r \frac{H(r, \rho)}{G(r, \alpha)} \chi(\rho)W(\rho)\, d\rho = 0.$$

Hence, by (9),

$$\int_\alpha^r \rho W(\rho)\, d\rho \geq \int_\alpha^r \frac{H(r, \rho)}{G(r, \alpha)} G(\rho, \alpha)W(\rho)\, d\rho \geq 2l + 1 - \theta(r) + \int_\alpha^r \frac{H(r, \rho)}{G(r, \alpha)} [G(\rho, \alpha) - \chi(\rho)]W(\rho)\, d\rho \quad (12)$$

We proceed as in case (*c*) above. The inequality $\int_\alpha^\infty \rho W(\rho)\, d\rho \geq 2l + 1$ is an immediate consequence of (12). To exclude equality the intervals $[\xi, \eta]$ and $[\eta, \zeta]$ are chosen as before $(\xi \geq \alpha)$, so that $(2l + 1)(G(\rho, \alpha) - \chi(\rho))\rho^{-(l+1)} \geq c' > 0$ if $\eta \leq \rho \leq \zeta$. For $r > \zeta$ (12) implies

$$\int_\alpha^r \rho W(\rho)\, d\rho \geq 2l + 1 - \theta(r) + c' \int_\eta^\zeta \rho^l \frac{H(r, \rho)}{H(r, \alpha)} \rho W(\rho)\, d\rho$$

and since $\lim_{r\to\infty}(H(r, \rho)/H(r, \alpha)) = (\alpha/\rho)^l$, we find for $r \to \infty$ $\int_\alpha^\infty \rho W(\rho)d\rho \geq 2l + 1 + c'\alpha^l \int_\eta^\zeta \rho W(\rho)\, d\rho > 2l + 1$. This concludes the proof of (6).

5. *Examples.*—The proofs in the preceding section suggest the construction of potentials for which the inequalities (3) or (6) may be approxi-

mately replaced by the corresponding equalities. In 4(*a*), for example, the first inequality will nearly reduce to an equality if at those r where $W(r)$ is appreciable $\phi(r)$ nearly equals r^{l+1}, i.e., the field free solution. This leads (for $n_l = 1$) to the choice of a potential $V(r) = -W(r)$ ($W \geq 0$) which vanishes everywhere with the exception of a small interval $a < r < a + \delta = b$. Outside $[a, b]$ we obtain for a suitably normalized solution of equation (5)

$$\left.\begin{aligned} \phi(r) &= (r/a)^{l+1} \qquad r \leq a \\ \phi(r) &= c_1(b/r)^l - c_2(r/b)^{l+1} \quad r \geq b \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} (2l+1)c_1 &= (l+1)\phi(b) - b\phi'(b); \\ (2l+1)c_2 &= -l\phi(b) - b\phi'(b) \end{aligned}\right\} \tag{14}$$

If $c_2 > 0$, $\phi(r) \to -\infty$ as $r \to \infty$, so that $\phi(r)$ vanishes at a point β given by $(\beta/b)^{2l+1} = c_1/c_2$, and if $c_2 = 0$, then $\beta = \infty$. Owing to the smallness of δ the relative change of $\phi(r)$ across the interval $[a, b]$ is negligible compared to the relative change of $\phi'(r)$, so that $\phi(b) \sim \phi(a)$. From the condition $c_2 \geq 0$ we obtain $-\phi'(b)/\phi(b) > l/b$, and since $\phi'(a)/\phi(a) = (l+1)/a$, this amounts to

$$\phi'(a)/\phi(a) - \phi'(b)/\phi(b) \sim (\phi'(a) - \phi'(b))/\phi(a) \gtrsim (2l+1)/a \tag{15}$$

Thus the required increment of the logarithmic derivative is the smaller the larger a is chosen—or a potential of given strength is the more effective in producing bound states the farther it is removed from the origin (which is the reason for the weight factor r in the integral I). Without yet specifying $W(r)$, we see from (5) that $\phi'(b) - \phi'(a)$ approximately equals $-\bar{W}\delta\phi(a)$ where $\bar{W}$ is a suitable average of W, provided the centrifugal term $l(l+1)r^{-2}$ is negligible compared to $\bar{W}$. If the increment is as small as possible we find from (15) that $\bar{W}\delta a \sim (2l+1)$ which is equivalent to $\int_a^b rW(r)\,dr = \int_0^\infty rW(r)\,dr \sim 2l+1$.

To have a definite example consider $W(r) = 1 + l(l+1)r^{-2}$, and $a = (2l+1)(1+\delta)/\delta$. Then, in $[a,b]$, $\phi(r) = \cos(r-a) + ((l+1)/a)\sin(r-\alpha)$ so that $\phi(b) = \cos\delta + ((l+1)/a)\sin\delta$, $\phi'(b) = -\sin\delta + ((l+1)/a)\cos\delta$, and one verifies easily that $c_2 > 0$ for small δ (e.g., $\delta < 1/4$). The zero, β, is determined by $(\beta/b)^{2l+1} = c_1/c_2$, and approximately $c_1/c_2 \sim \delta^{-1}$ so that $\beta \sim a \cdot \delta^{-1/(2l+1)}$. Finally, $I = \int_a^b rW(r)\,dr = \delta(a + {}^1/_2\delta) + l(l+1)\log(1+\delta/a)$. As $\delta \to 0$, $I \to 2l+1$. Alternately, instead of varying a and keeping the strength of W fixed, one might keep a fixed and vary the strength of the potential.

In a similar way one may construct potentials with two or more bound state solutions such that I is arbitrarily close to $(2l+1)n_l$. One simply has to add other troughs in suitably placed intervals $[a', a' + \delta']$, etc., in such a way, however, that two successive intervals are sufficiently far

from one another and from the zero of $\phi(r)$ between them. Note that these potentials are adjusted only to one fixed value of the angular momentum.

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[5] For $l = 0$ these statements are proved in the appendix of ref. 4. For higher l a similar proof may be given.

[6] Even for $l = 0$ this case is significant although no bound state is present. In particular in this case $|\sin \eta(0)| = 1$ where $\eta(k)$ is the scattering phase shift, and hence the cross section $\sin^2 \eta(k)/k^2$ becomes infinite as $k \to 0$. (See ref. 4, equation (1.9). In this case $f(0) = 0$.)

[7] For $l = 0$ one need not distinguish $\alpha = 0$ and $\alpha > 0$, because $G(r, \alpha) = r - \alpha$ may be used in both cases.

ON THE INVARIANT THEORY OF THE CLASSICAL GROUPS

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It has been recognized for some time that the theory of invariants and covariants, with respect to a given group, rests on the analysis of $\Gamma \otimes \Gamma'$, where Γ and Γ' are irreducible representations of the group, into its irreducible components. Thus if we denote by $\{\lambda\}$ the irreducible representation of the n-dimensional linear group which is associated with the partition $(\lambda) = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$, of any non-negative integer m into not more than n parts the core of the theory of invariants and covariants, under linear transformations, is the analysis of $\{\lambda\} \otimes \{\mu\}$ where (λ) and (μ) are partitions of any two non-negative integers m and j, respectively. The cases where (λ) is either the 1-element partition (m) or the m-element partition (1^m) and (μ) is either the 1-element partition (j) or the j-element partition (1^j) are of particular importance and the problem of analyzing $\{\lambda\} \otimes \{\mu\}$, especially in these cases, has been much studied, following the initial impetus given by Littlewood,[1] during the past decade. However the methods used have been laborious when m and j are greater than 2; in these cases $\{\lambda\} \otimes \{\mu\}$ contains many components, each corresponding to a partition of mj, and each of these has had to be determined separately by a tedious calculation. We present in this note a method which yields, in the cases of particular importance referred to, the components of $\{\lambda\} \otimes \{\mu\}$ in platoons, rather than individually, each platoon consisting of those parentheses $\{\ldots\}$ which contain the same number of non-zero parts.

We also furnish some general formulas for the analyses of $[\lambda] \otimes \{\mu\}$ and of $\langle\lambda\rangle \otimes \{\mu\}$ where $[\lambda]$ and $\langle\lambda\rangle$ are irreducible representations of the n-dimensional orthogonal group and of the $2k$-dimensional symplectic group, respectively, and we point out a connection between $[\lambda]$ and $\langle\lambda\rangle$ which seems not to have been noticed and which enables us to write down without calculation the analysis of $\langle\lambda\rangle \otimes \{\mu\}$ when the analysis of $[\lambda^*] \otimes \{\mu\}$, or of $[\lambda^*] \otimes \{\mu^*\}$, is known, and vice versa (the $*$ denoting, as usual, the associated partition).

Neither $\{m\} \otimes \{j\}$ nor $\{m\} \otimes \{1^j\}$ contains components corresponding to parentheses $\{\ldots\}$ with more than j non-zero elements or, as we shall say for brevity, terms with more than j elements, and the j-element terms of these are obtained by adding 1 to each element of each term of the analyses A_j and A_j' of $\{m-1\} \otimes \{1^j\}$ and $\{m-1\} \otimes \{j\}$, respectively, each term of these analyses being written as a j-element term by the addition of zeros, if necessary. For example, the 3-element terms of $\{3\} \otimes \{3\}$ are $\{52^2\} + \{4^21\}$ since $\{2\} \otimes \{1^3\} = \{41^2\} + \{3^2\}$ and the 3-element terms of $\{3\} \otimes \{1^3\}$ are $\{71^2\} + \{531\} + \{3^3\}$ since $\{2\} \otimes \{3\} = \{6\} + \{42\} + \{2^3\}$. To obtain the $(j-1)$-element terms of $\{m\} \otimes \{j\}$ we subtract $A_j\{1\}$ from $(\{m-1\} \otimes \{1^{j-1}\})\{m\}$, where we may simplify the calculations involved by assuming that $n = j-1$, so that all j-element terms vanish. We refer to this hypothesis by saying that we make a $(j-1)$-ary analysis, A_{j-1}, of $(\{m-1\} \otimes \{1^{j-1}\})\{m\} - A_j\{1\}$. Then the $(j-1)$-element terms of $\{m\} \otimes \{j\}$ are obtained by adding 1 to each element of each term of A_{j-1}, each of these terms being written as a $(j-1)$-element term. Similarly the $(j-1)$-element terms of $(\{m\} \otimes \{1^j\})$ are obtained by adding 1 to each element of each term of the $(j-1)$-ary analysis, A_{j-1}', of $(\{m-1\} \otimes \{j-1\})\{m\} - A_j'\{1\}$. Since $\{2\} \otimes \{1^2\} = \{31\}$, and since $\{31\} = \{1^2\}\{2\}$ when $n = 2$, A_{j-1} is, when $m = 3$ and $j = 3$, $A_2 = \{1^2\}\{2\}\{3\} - \{3^2\}\{1\} = \{61\} + \{52\}$ and, similarly, A_{j-1}' is $A_2' = (\{4\} + \{2^2\})\{3\} - (\{6\} + \{42\})\{1\} = \{52\}$. Thus the 2-element terms of $\{3\} \otimes \{3\}$ are $\{72\} + \{63\}$ and the 2-element term of $\{3\} \otimes \{1^3\}$ is $\{63\}$. To obtain the $(j-2)$-element terms of $\{m\} \otimes \{j\}$ we make a $(j-2)$-ary analysis, A_{j-2}, of $(\{m-1\} \otimes \{1^{j-2}\})(\{m\} \otimes \{2\}) - A_{j-1}\{1\} - A_j\{1^2\}$ and we increase each element of each term of A_{j-2} by 1. Similarly to obtain the $(j-2)$-element terms of $\{m\} \otimes \{1^j\}$ we perform the $(j-2)$-ary analysis, A_{j-2}', of $(\{m-1\} \otimes \{j-2\})(\{m\} \otimes \{1^2\}) - A_{j-1}'\{1\} - A_j'\{1^2\}$ and add 1 to each element of each term of the result. Thus the 1-element term of $\{3\} \otimes \{3\}$ is $\{9\}$ and $\{3\} \otimes \{1^3\}$ contains no 1-element term. This completes the analyses of $\{3\} \otimes \{3\}$ and of $\{3\} \otimes \{1^3\}$: $\{3\} \otimes \{3\} = \{9\} + \{72\} + \{63\} + \{52^2\} + \{4^21\}$; $\{3\} \otimes \{1^3\} = \{71^2\} + \{63\} + \{531\} + \{3^3\}$. When $n < 3$ these formulas must be modified by means of the *modification rule* for the linear group which is simply to drop all terms with more than n elements. Thus, for the plane linear group, for which

$n = 2$, we have $\{3\} \otimes \{3\} = \{9\} + \{72\} + \{63\}$; $\{3\} \otimes \{1^3\} = \{63\}$. When $j > 3$ the method continues in the same way; to obtain the $(j-3)$-element terms of $\{m\} \otimes \{j\}$ we perform the $(j-3)$-ary analysis, A_{j-3}, of $(\{m-1\} \otimes \{1^{j-3}\})(\{m\} \otimes \{3\}) - A_{j-2}\{1\} - A_{j-1}\{1^2\} - A_j\{1^3\}$ and add 1 to each element of each term of the result; to obtain the $(j-3)$-element terms of $\{m\} \otimes \{1^j\}$ we perform the $(j-3)$-ary analysis, A'_{j-3}, of $(\{m-1\} \otimes \{j-3\})(\{m\} \otimes \{1^3\}) - A'_{j-2}\{1\} - A'_{j-1}\{1^2\} - A'_j\{1^3\}$ and add 1 to each element of each term of the result; and so on. For example the 4-element terms of $\{3\} \otimes \{1^4\}$ are $\{91^3\} + \{731^2\} + \{5^21^2\} + \{53^21\} + \{3^4\}$; the 3-element terms are $\{831\} + \{741\} + \{642\} + \{63^2\}$; there is only one 2-element term $\{6^2\}$ and there is no 1-element term so that $\{3\} \otimes \{1^4\} = \{91^3\} + \{831\} + \{741\} + \{731^2\} + \{6^2\} + \{642\} + \{63^2\} + \{5^21^2\} + \{53^21\} + \{3^4\}$. For the 3-dimensional linear group $\{3\} \otimes \{1^4\} = \{831\} + \{741\} + \{6^2\} + \{642\} + \{63^2\}$ and for the plane linear group $\{3\} \otimes \{1^4\} = \{6^2\}$.

It is clear, then, that the method we propose here for analyzing $\{m\} \otimes \{j\}$ and $\{m\} \otimes \{1^j\}$ is a recurrence one, the analysis of $\{m\} \otimes \{j\}$ depending on a knowledge of the analyses of $\{m-1\} \otimes \{1^j\}$ and of $\{m\} \otimes \{k\}$ for $k < j$. Thus both problems, namely, the problems of analyzing $\{m\} \otimes \{j\}$ and $\{m\} \otimes \{1^j\}$, are carried out together. In particular, the determination of the j-element terms of $\{m\} \otimes \{j\}$ depends on the analysis of $\{m-1\} \otimes \{1^j\}$, rather than on the analysis of $\{m-1\} \otimes \{j\}$, and the determination of the j-element terms of $\{m\} \otimes \{1^j\}$ depends on the analysis of $\{m-1\} \otimes \{j\}$, rather than on the analysis of $\{m-1\} \otimes \{1^j\}$. It is, doubtless, this "crossing over" procedure that has hitherto masked the ease with which the j-element terms (i.e., the terms with the greatest number of elements) of $\{m\} \otimes \{j\}$ and of $\{m\} \otimes \{1^j\}$ may be determined.

Turning now to the n-dimensional orthogonal group (where $n = 2k$, if n is even, and $n = 2k + 1$, if n is odd) there is associated with each set (λ) of not more than k, rather than n, non-negative integers an irreducible representation $[\lambda]$ of the group. $[\lambda]$ is a generalized representation (i.e., a linear combination with integral coefficients, possibly negative, of irreducible representations) of the containing n-dimensional linear group so that the problem of analyzing the product $[\lambda][\mu]$ depends on the corresponding problem for the linear group (i.e., the problem of analyzing $\{\lambda\}\{\mu\}$) and on the modification rule for the orthogonal group (for we encounter, in the analysis of $\{\lambda\}\{\mu\}$, terms $\{\ldots\}$ containing more than k elements and these introduce terms $[\ldots]$ containing more than k elements). We owe to Newell[2] both the solution of the problem of analyzing $[\lambda][\mu]$ and the modification rule for the orthogonal group. When (λ) and (μ) are 1-element partitions (m) and (j) the product $\{m\}\{j\}$ is $\{m+j\} + \{m+j-1, 1\} + \ldots$ the sum ending with $\{m, j\}$, where we assume, without any lack of generality, that $m \geq j$. Newell has shown that $[m][j]$ is $[\{m\}\{j\} +$

$\{m-1\}\{j-1\}+\ldots+\{m-j\}]$, the enclosure of any linear combination of the parentheses $\{\ldots\}$ in square parentheses indicating that each $\{\ldots\}$ is to be replaced by the corresponding $[\ldots]$. For example, $[3][2] = [\{3\}\{2\}+\{2\}\{1\}+\{1\}] = [5]+[41]+[32]+[3]+[21]+[1]$. For the 3-dimensional orthogonal group the modification rule states that all 2-element parentheses $[\lambda_1\lambda_2]$, save those for which $\lambda_2 = 1$, vanish while $[\lambda 1] = [\lambda]'$, where $[\lambda]'$ is the product of $[\lambda]$ by the 1-dimensional alternating, or determinant, representation which is 1 over the rotation half, and -1 over the reflexion half, of the group. In this way we recover the Clebsch-Gordan formula $[3][2] = [5]+[4]'+[3]+[2]'+[1]$. It follows readily from the analysis of $[m][m]$ that $[m]\otimes\{2\} = [(\{m\}\otimes\{2\})+(\{m-1\}\otimes\{2\})+\ldots+\{0\}]$ and that $[m]\otimes\{1^2\} = [(\{m\}\otimes\{1^2\})+(\{m-1\}\otimes\{1^2\})+\ldots+\{1^2\}]$. For example

$$[2]\otimes\{2\} = [4]+[2^2]+[2]+[0];\quad [2]\otimes\{1^2\} = [31]+[1^2];$$
$$[3]\otimes\{2\} = [6]+[42]+[4]+[2^2]+[2]+[0];\quad [3]\otimes\{1^2\} = [51]+[3^2]+[31]+[1^2];$$
$$[4]\otimes\{2\} = [8]+[62]+[4^2]+[6]+[42]+[4]+[2^2]+[2]+[0];$$
$$[4]\otimes\{1^2\} = [71]+[53]+[51]+[3^2]+[31]+[1^2].$$

For the 3-dimensional orthogonal group these modify to

$$[2]\otimes\{2\} = [4]+[2]+[0];\quad [2]\otimes\{1^2\} = [3]'+[1]';$$
$$[3]\otimes\{2\} = [6]+[4]+[2]+[0];\quad [3]\otimes\{1^2\} = [5]'+[3]'+[1]'$$

and so on. In general we have for the 3-dimensional orthogonal group

$$[m]\otimes\{2\} = [2m]+[2m-2]+\ldots+[0];\quad [m]\otimes\{1^2\} = [2m-1]'+[2m-3]'+\ldots+[1]'.$$

The representations $[m]\otimes\{j\}$ and $[m]\otimes\{1^j\}$ of the n-dimensional orthogonal group may be analyzed by means of the recurrence formulas

$$[m]\otimes\{j\} = (\{m\}\otimes\{j\})-([m]\otimes\{j-1\})\{m-2\}-([m]\otimes\{j-2\})(\{m-2\}\otimes\{2\})-\ldots-(\{m-2\}\otimes\{j\})$$
$$[m]\otimes\{1^j\} = (\{m\}\otimes\{1^j\})-([m]\otimes\{1^{j-1}\})\{m-2\}-([m]\otimes\{1^{j-2}\})(\{m-2\}\otimes\{1^2\})-\ldots-(\{m-2\}\otimes\{1^j\})$$

and a known formula which furnishes the analysis of the representation induced over the orthogonal group by any irreducible representation of the containing n-dimensional linear group. In this way we obtain the following analyses:

$$[2]\otimes\{3\} = [6]+[42]+[2^3]+[4]+[31]+[2^2]+2[2]+[0];$$
$$[2]\otimes\{1^3\} = [41^2]+[3^2]+[31]+[21^2]+[1^2];$$
$$[2]\otimes\{4\} = [8]+[62]+[4^2]+[42^2]+[2^4]+[6]+[51]+2[42]+[321]+[2^3]+3[4]+[31]+3[2^2]+3[2]+2[0];$$

$$[2] \otimes \{1^4\} = [51^3] + [431] + [42] + [41^2] + [321] + [31^3] + [31] + [2^2] + [21^2] + [1^4] + [2];$$
$$[3] \otimes \{3\} = [9] + [72] + [63] + [52^2] + [4^21] + [7] + [61] + 2[52] + [43] + [421] + [32^2] + 2[5] + 2[41] + 2[32] + [2^21] + 3[3] + [21] + [1];$$
$$[3] \otimes \{1^3\} = [71^2] + [63] + [531] + [3^3] + [61] + [52] + [51^2] + [43] + [421] + [3^21] + 2[41] + [32] + 2[31^2] + [3] + [21] + [1^3].$$

These modify, for the 3-dimensional orthogonal group, to

$$[2] \otimes \{3\} = [6] + [4] + [3]' + [2] + [0]; \quad [2] \otimes \{1^3\} = [3]' + [1]';$$
$$[2] \otimes \{4\} = [8] + [6] + [5]' + 2[4] + 2[2] + [0]; \quad [2] \otimes \{1^4\} = [2];$$
$$[3] \otimes \{3\} = [9] + [7] + [6]' + [5] + [4]' + 2[3] + [1]; \quad [3] \otimes \{1^3\} = [6]' + [4]' + [3] + [2]' + [0]'.$$

For the 4-dimensional orthogonal group they modify to

$$[2] \otimes \{3\} = [6] + [42] + [4] + [31] + 2[2] + [0]; \quad [2] \otimes \{1^3\} = [3^2] + [4]' + [31] + [2]' + [1^2];$$
$$[2] \otimes \{4\} = [8] + [62] + [4^2] + [6] + [51] + [42] + 3[4] + [31] + 2[2^2] + 2[2] + 2[0];$$
$$[2] \otimes \{1^4\} = [42] + [4]' + [31] + [2^2] + [2] + [2]' + [0]';$$
$$[3] \otimes \{3\} = [9] + [72] + [63] + [7] + [61] + [52] + [43] + 2[5] + 2[41] + [32] + 3[3] + [21] + [1];$$
$$[3] \otimes \{1^3\} = [63] + [7]' + [61] + [52] + [43] + [5]' + 2[41] + [32] + [3] + 2[3]' + [21] + [1]'.$$

For the 5-dimensional orthogonal group they modify to

$$[2] \otimes \{3\} = [6] + [42] + [4] + [31] + [2^2] + 2[2] + [0]; \quad [2] \otimes \{1^3\} = [41]' + [3^2] + [31] + [21]' + [1^2];$$
$$[2] \otimes \{4\} = [8] + [62] + [4^2] + [6] + [51] + 2[42] + [32]' + 3[4] + [31] + 2[2^2] + 3[2] + 2[0];$$
$$[2] \otimes \{1^4\} = [43]' + [42] + [5]' + [41]' + [32]' + [31] + [2^2] + [3]' + [21]' + [2] + [1]';$$
$$[3] \otimes \{3\} = [9] + [72] + [63] + [4^2]' + [7] + [61] + 2[52] + [43] + [42]' + 2[5] + 2[41] + 2[32] + [2^2]' + 3[3] + [21] + [1];$$
$$[3] \otimes \{1^3\} = [63] + [71]' + [53]' + [61] + [52] + [43] + [51]' + [42]' + [3^2]' + 2[41] + [32] + 2[31]' + [3] + [21] + [1^2]'.$$

For the 6-dimensional orthogonal group the formula for $[2] \otimes \{4\}$ modifies to

$$[2] \otimes \{4\} = [8] + [62] + [4^2] + [42^2] + [6] + [51] + 2[42] + [321] + 3[4] + [31] + 3[2^2] + 3[2] + 2[0]$$

and the formula for $[2] \otimes \{1^4\}$ modifies to

$$[2] \otimes \{1^4\} = [431] + [51]' + [42] + [41^2] + [321] + [31] + [31]' + [2^2] + [21^2] + [2] + [1^2]'$$

while for the 7-dimensional orthogonal group they modify to

$$[2] \otimes \{4\} = [8] + [62] + [4^2] + [42^2] + [6] + [51] + 2[42] + [321] + [2^3] + 3[4] + [31] + 3[2^2] + 3[2] + 2[0];$$
$$[2] \otimes \{1^4\} = [431] + [51^2]' + [42] + [41^2] + [321] + [31^2]' + [31] + [2^2] + [21^2] + [1^3]' + [2].$$

If $\Gamma = \sum_{(\lambda)} c_{(\lambda)} \{\lambda\}$ is any generalized representation of the n-dimensional linear group we denote by Γ^* the associated generalized representation $\sum_{(\lambda)} c_{(\lambda)} \{\lambda^*\}$ where (λ^*) is the associated, or conjugate, partition to (λ). $[\lambda]$ is a generalized representation of the linear group and the irreducible representations $\langle\lambda\rangle$ of the $n = 2k$-dimensional unitary symplectic group are connected with the irreducible representations $[\lambda]$ of the $2k$-dimensional orthogonal group by means of the relation $\langle\lambda\rangle = [\lambda^*]^*$. If, then, (λ) is a partition of an even integer m, $\langle\lambda\rangle \otimes \{\mu\} = ([\lambda^*] \otimes \{\mu\})^*$ while, if (λ) is a partition of odd integer m, $\langle\lambda\rangle \otimes \{\mu\} = ([\lambda^*] \otimes \{\mu^*\})^*$. These relations enable us to write down, without any calculation, the analysis of $\langle\lambda\rangle \otimes \{\mu\}$ when the analyses of $[\lambda^*] \otimes \{\mu\}$ and of $[\lambda^*] \otimes \{\mu^*\}$ are known. Thus we have the formulas

$$\langle 1^m \rangle \otimes \{2\} = \langle((\{1^m\} \otimes \{2\}) + (\{1^{m-1}\} \otimes \{1^2\}) + \ldots)\rangle$$

(the series ending in $\{0\}$ if m is even and in $\{2\}$ if m is odd) and

$$\langle 1^m \rangle \otimes \{1^2\} = \langle((\{1^m\} \otimes \{1^2\}) + (\{1^{m-1}\} \otimes \{2\}) + \ldots)\rangle$$

(the series ending in $\{2\}$ if m is even and in $\{0\}$ if m is odd). For example

$$\langle 1^2 \rangle \otimes \{2\} = \langle 2^2 \rangle + \langle 1^4 \rangle + \langle 1^2 \rangle + \langle 0 \rangle; \quad \langle 1^2 \rangle \otimes \{1^2\} = \langle 21^2 \rangle + \langle 2 \rangle;$$
$$\langle 1^3 \rangle \otimes \{2\} = \langle 2^3 \rangle + \langle 21^4 \rangle + \langle 21^2 \rangle + \langle 2 \rangle;$$
$$\langle 1^3 \rangle \otimes \{1^2\} = \langle 2^2 1^2 \rangle + \langle 1^6 \rangle + \langle 2^2 \rangle + \langle 1^4 \rangle + \langle 1^2 \rangle + \langle 0 \rangle$$

and so on. Similarly

$$\langle 2 \rangle \otimes \{2\} = \langle 4 \rangle + \langle 2^2 \rangle + \langle 1^2 \rangle + \langle 0 \rangle; \quad \langle 2 \rangle \otimes \{1^2\} = \langle 31 \rangle + \langle 2 \rangle;$$
$$\langle 3 \rangle \otimes \{2\} = \langle 6 \rangle + \langle 42 \rangle + \langle 31 \rangle + \langle 2 \rangle;$$
$$\langle 3 \rangle \otimes \{1^2\} = \langle 51 \rangle + \langle 3^2 \rangle + \langle 4 \rangle + \langle 2^2 \rangle + \langle 1^2 \rangle + \langle 0 \rangle;$$
$$\langle 2 \rangle \otimes \{3\} = \langle 6 \rangle + \langle 42 \rangle + \langle 2^3 \rangle + \langle 31 \rangle + \langle 21^2 \rangle + 2\langle 2 \rangle;$$
$$\langle 2 \rangle \otimes \{1^3\} = \langle 3^2 \rangle + \langle 41^2 \rangle + \langle 4 \rangle + \langle 31 \rangle + \langle 2^2 \rangle + \langle 1^2 \rangle + \langle 0 \rangle.$$

$$\langle 1^2\rangle \otimes \{3\} = \langle 3^2\rangle + \langle 2^2 1^2\rangle + \langle 1^6\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^4\rangle + 2\langle 1^2\rangle + \langle 0\rangle;$$

$$\langle 1^2\rangle \otimes \{1^3\} = \langle 31^3\rangle + \langle 2^3\rangle + \langle 31\rangle + \langle 21^2\rangle + \langle 2\rangle;$$

$$\langle 2\rangle \otimes \{4\} = \langle 8\rangle + \langle 62\rangle + \langle 4^2\rangle + \langle 42^2\rangle + \langle 2^4\rangle + \langle 51\rangle + \langle 41^2\rangle + \langle 3^2\rangle + \langle 321\rangle + \langle 2^2 1^2\rangle + 2\langle 4\rangle + \langle 31\rangle + 3\langle 2^2\rangle + \langle 1^4\rangle + 2\langle 1^2\rangle + 2\langle 0\rangle;$$

$$\langle 2\rangle \otimes \{1^4\} = \langle 51^3\rangle + \langle 431\rangle + \langle 51\rangle + \langle 42\rangle + \langle 41^2\rangle + \langle 3^2\rangle + \langle 321\rangle + \langle 4\rangle + \langle 31\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 2\rangle + \langle 1^2\rangle;$$

$$\langle 1^2\rangle \otimes \{4\} = \langle 4^2\rangle + \langle 3^2 1^2\rangle + \langle 2^4\rangle + \langle 2^2 1^4\rangle + \langle 1^8\rangle + \langle 3^2\rangle + \langle 321\rangle + 2\langle 2^2 1^2\rangle + \langle 21^4\rangle + \langle 1^6\rangle + 3\langle 2^2\rangle + \langle 21^2\rangle + 3\langle 1^4\rangle + 3\langle 1^2\rangle + 2\langle 0\rangle;$$

$$\langle 1^2\rangle \otimes \{1^4\} = \langle 41^4\rangle + \langle 32^2 1\rangle + \langle 41^2\rangle + \langle 321\rangle + \langle 31^3\rangle + \langle 2^2 1^2\rangle + \langle 4\rangle + \langle 31\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^2\rangle.$$

For the 2-dimensional unitary symplectic group (i.e., the 2-dimensional unimodular unitary group) these modify as follows:

$$\langle 2\rangle \otimes \{2\} = \langle 4\rangle + \langle 0\rangle;\ \langle 2\rangle \otimes \{1^2\} = \langle 2\rangle;\ \langle 3\rangle \otimes \{2\} = \langle 6\rangle + \langle 2\rangle;\ \langle 3\rangle \otimes \{1^2\} = \langle 4\rangle + \langle 0\rangle;$$

$$\langle 2\rangle \otimes \{3\} = \langle 6\rangle + \langle 2\rangle;\ \langle 2\rangle \otimes \{1^3\} = \langle 0\rangle;\ \langle 2\rangle \otimes \{4\} = \langle 8\rangle + \langle 4\rangle + \langle 0\rangle.$$

For the 4-dimensional unitary symplectic group the necessary modifications are

$$\langle 1^2\rangle \otimes \{2\} = \langle 2^2\rangle + \langle 0\rangle;\ \langle 1^2\rangle \otimes \{1^2\} = \langle 2\rangle;\ \langle 2\rangle \otimes \{3\} = \langle 6\rangle + \langle 42\rangle + \langle 31\rangle + 2\langle 2\rangle;$$

$$\langle 2\rangle \otimes \{1^3\} = \langle 3^2\rangle + \langle 4\rangle + \langle 31\rangle + \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle;\ \langle 1^2\rangle \otimes \{3\} = \langle 3^2\rangle + \langle 1^2\rangle;\ \langle 1^2\rangle \otimes \{1^3\} = \langle 2\rangle;$$

$$\langle 2\rangle \otimes \{4\} = \langle 8\rangle + \langle 62\rangle + \langle 4^2\rangle + \langle 51\rangle + \langle 3^2\rangle + 2\langle 4\rangle + \langle 31\rangle + 2\langle 2^2\rangle + \langle 1^2\rangle + 2\langle 0\rangle;$$

$$\langle 2\rangle \otimes \{1^4\} = \langle 42\rangle + \langle 3^2\rangle + \langle 4\rangle + \langle 31\rangle + \langle 2^2\rangle + \langle 2\rangle + \langle 1^2\rangle;$$

$$\langle 1^2\rangle \otimes \{4\} = \langle 4^2\rangle + \langle 2^2\rangle + \langle 0\rangle;\ \langle 1^2\rangle \otimes \{1^4\} = \langle 1^2\rangle$$

while for the 6-dimensional unitary symplectic group the necessary modifications are

$$\langle 1^2\rangle \otimes \{2\} = \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle;\ \langle 1^3\rangle \otimes \{2\} = \langle 2^3\rangle + \langle 2\rangle;\ \langle 1^3\rangle \otimes \{1^2\} = \langle 2^2\rangle + \langle 0\rangle;$$

$$\langle 1^2\rangle \otimes \{3\} = \langle 3^2\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^2\rangle + \langle 0\rangle;\ \langle 1^2\rangle \otimes \{1^3\} = \langle 2^3\rangle + \langle 31\rangle + \langle 21^2\rangle + \langle 2\rangle$$

$$\langle 2\rangle \otimes \{4\} = \langle 8\rangle + \langle 62\rangle + \langle 4^2\rangle + \langle 42^2\rangle + \langle 51\rangle + \langle 41^2\rangle + \langle 3^2\rangle + \langle 321\rangle + 2\langle 4\rangle + \langle 31\rangle + 3\langle 2^2\rangle + 2\langle 1^2\rangle + 2\langle 0\rangle;$$

$$\begin{aligned}
\langle 2\rangle \otimes \{1^4\} &= \langle 431\rangle + \langle 51\rangle + \langle 42\rangle + \langle 41^2\rangle + \langle 3^2\rangle + \langle 321\rangle \\
&\quad + \langle 4\rangle + \langle 31\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 2\rangle + \langle 1^2\rangle; \\
\langle 1^2\rangle \otimes \{4\} &= \langle 4^2\rangle + \langle 3^2\rangle + \langle 321\rangle + 2\langle 2^2\rangle + 2\langle 1^2\rangle + \langle 0\rangle; \\
\langle 1^2\rangle \otimes \{1^4\} &= \langle 321\rangle + \langle 4\rangle + \langle 31\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^2\rangle.
\end{aligned}$$

For the 8-dimensional unitary symplectic group the necessary modifications are

$$\begin{aligned}
\langle 1^3\rangle \otimes \{2\} &= \langle 2^3\rangle + \langle 21^2\rangle + \langle 2\rangle; \quad \langle 1^3\rangle \otimes \{1^2\} = \langle 2^21^2\rangle + \\
&\qquad \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle; \\
\langle 1^2\rangle \otimes \{3\} &= \langle 3^2\rangle + \langle 2^21^2\rangle + \langle 2^2\rangle + \langle 21^2\rangle + 2\langle 1^2\rangle + \langle 0\rangle; \\
\langle 1^2\rangle \otimes \{4\} &= \langle 4^2\rangle + \langle 3^21^2\rangle + \langle 2^4\rangle + \langle 3^2\rangle + \langle 321\rangle + \\
&\qquad \langle 2^21^2\rangle + 3\langle 2^2\rangle + 2\langle 1^4\rangle + 2\langle 1^2\rangle + 2\langle 0\rangle; \\
\langle 1^2\rangle \otimes \{1^4\} &= \langle 32^21\rangle + \langle 41^2\rangle + \langle 321\rangle + \langle 31^3\rangle + \langle 2^21^2\rangle + \\
&\qquad \langle 4\rangle + \langle 31\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^2\rangle;
\end{aligned}$$

and so on.

[1] Littlewood, D. E., *Phil. Trans. Roy. Soc.*, **A, 239,** 305–365, 387–417 (1944).
[2] Newell, M. J., *Proc. Roy. Irish Acad.*, **54A,** 143–163 (1951).

ON THE DECOMPOSITION OF TENSORS BY CONTRACTION

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In a paper with the above title, published in the Einstein Jubilee number of *Reviews of Modern Physics*, Racah[1] has furnished formulae which enable us to tell how the irreducible representations of the 3- and 4-dimensional linear groups reduce over their rotation subgroups. These formulas furnish separately the number of times that any individual irreducible representation of the rotation group (either 3- or 4-dimensional) appears in the reduction of the given irreducible representation of the containing linear group. We show in the present note how to obtain the desired reduction with very little labor (the components being obtained in groups, rather than one by one); more than this, our method furnishes the reduction not only over the rotation subgroup but also over the orthogonal subgroup and may be applied not only to the 3- and 4-dimensional orthogonal groups but to orthogonal groups of any dimension. Racah's method was developed for the 3-dimensional rotation group and its application to the 4-dimensional rotation group depends on a fortuitous factorization of the latter. In view of possible physical applications of the 5-dimensional orthog-

onal group it seems desirable to publish this alternative to Racah's method.

We denote by $\{\lambda\} = \{\lambda_1, \ldots, \lambda_n\}$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$, the typical irreducible representation of the n-dimensional linear group, by $[\lambda] = [\lambda_1, \ldots, \lambda_k]$, where $n = 2k$ or $2k + 1$, the typical irreducible representation of the n-dimensional orthogonal group and by ϵ the determinant of a typical matrix of this orthogonal group (so that $\epsilon = 1$ over the rotation half and $\epsilon = -1$ over the reflexion half, of the group). Note that $\epsilon^j = 1$, if j is even, and $\epsilon^j = \epsilon$ if j is odd. The product of $[\lambda]$ by the alternating, or determinant, representation of the orthogonal group is an irreducible representation which we denote by $\epsilon[\lambda]$. $(\lambda) = (\lambda_1, \ldots, \lambda_n)$ is a partition of a non-negative integer m: $\lambda_1 + \lambda_2 + \ldots + \lambda_n = m$. If $(\lambda) = (\lambda_1)$ contains only one non-zero part we have, for every n, the reduction

$$\{\lambda_1\} = [\lambda_1] + [\lambda_1 - 2] + \ldots.$$

the sum ending with $[0]$ if λ_1 is even and with $[1]$ if λ_1 is odd. The various terms of this analysis are of degrees m, $m - 2$, $m - 4$ and so on. If $(\lambda) = (\lambda_1, \lambda_2)$ contains only two non-zero parts we have a similar analysis into groups of terms of degrees m, $m - 2$, ... There is one term of degree m, $[\lambda_1, \lambda_2]$; there are 3 terms of degree $m - 2$, $[\lambda_1 - 2, \lambda_2] + [\lambda_1 - 1, \lambda_2 - 1] + [\lambda_1, \lambda_2 - 2]$; there are 5 terms of degree $m - 4$, $[\lambda_1 - 4, \lambda_2] + [\lambda_1 - 3, \lambda_2 - 1] + 2[\lambda_1 - 2, \lambda_2 - 2] + [\lambda_1 - 1, \lambda_2 - 3] + [\lambda_1, \lambda_2 - 4]$; there are 7 terms of degree $m - 6$, $[\lambda_1 - 6, \lambda_2] + [\lambda_1 - 5, \lambda_2 - 1] + 2[\lambda_1 - 4, \lambda_2 - 2] + 2[\lambda_1 - 3, \lambda_2 - 3] + 2[\lambda_1 - 2, \lambda_2 - 4] + [\lambda_1 - 1, \lambda_2 - 5] + [\lambda_1, \lambda_2 - 6]$; there are 9 terms of degree $m - 8$, $[\lambda_1 - 8, \lambda_2] + [\lambda_1 - 7, \lambda_2 - 1] + 2[\lambda_1 - 6, \lambda_2 - 2] + 2[\lambda_1 - 5, \lambda_2 - 3] + 3[\lambda_1 - 4, \lambda_2 - 4] + 2[\lambda_1 - 3, \lambda_2 - 5] + 2[\lambda_1 - 2, \lambda_2 - 6] + [\lambda_1 - 1, \lambda_2 - 7] + [\lambda_2 - 8]$, and so on. The key to the determination of these various groups of terms lies in expressing $\{2\} \otimes \{j\}$, $j = 1, 2, 3 \ldots$, in terms of the characteristic numbers ξ_1 and ξ_2 of a typical element of the 2-dimensional linear group. For example, $\{2\} \otimes \{1\} = \{2\} = \xi_1^2 + \xi_1\xi_2 + \xi_2^2$ which yields the 3 terms of degree $m - 2$; $\{2\} \otimes \{2\} = \{4\} + \{2^2\} = \xi_1^4 + \xi_1^3\xi_2 + 2\xi_1^2\xi_2^2 + \xi_1\xi_2^3 + \xi_2^4$ which yields the 5 terms of degree $m - 4$; $\{2\} \otimes \{3\} = \{6\} + \{42\}$ (for the 2-dimensional linear group) $= \xi_1^6 + \xi_1^5\xi_2 + 2\xi_1^4\xi_2^2 + 2\xi_1^3\xi_2^3 + 2\xi_1^2\xi_2^4 + \xi_1\xi_2^5 + \xi_2^6$ which yields the 7 terms of degree $m - 6$; and so on. When $(\lambda) = (\lambda_1, \lambda_2, \lambda_3)$ contains more than two non-zero parts this method for determining the reduction of $\{\lambda\}$ over the orthogonal group becomes cumbersome and a more convenient method is available,[2] the details of which we need not enter into here.

For the 3-dimensional orthogonal group it is unnecessary to consider the case where (λ) contains three non-zero parts since, over the orthogonal subgroup, $\{\lambda_1, \lambda_2, \lambda_3\} = \epsilon^{\lambda_3}\{\lambda_1 - \lambda_3, \lambda_2 - \lambda_3\}$. Moreover $[\lambda_1, \lambda_2]$ vanishes

if $\lambda_2 > 1$ and $[\lambda_1, 1] = \epsilon[\lambda_1]$. The calculations are further simplified by the following facts:

(1) $\{\lambda_1, \lambda_2\} = \epsilon^{\lambda_1}\{\lambda_1, \lambda_1 - \lambda_2\}$ over the orthogonal subgroup. In particular $\{2j, 2j\} = \{2j\} = [2j] + [2j-2] + \ldots + [0]$ and $\{2j+1, 2j+1\} = \epsilon\{2j+1\} = \epsilon[2j+1] + \epsilon[2j-1] + \ldots + \epsilon[1]$.

(2) $\{j+1, j\} = \epsilon^j[j+1] + \epsilon^{j-1}[j] + \ldots + [1]$.

(3) $\{\lambda_1, \lambda_2\} = \{\lambda_1 - 2, \lambda_2\} + \epsilon^{\lambda_2}[\lambda_1] + \epsilon^{\lambda_2 - 1}[\lambda_1 - 1] + \ldots + [\lambda_1 - \lambda_2]$.

These facts enable us to write down, with practically no labor of calculation, the reduction of $\{\lambda_1, \lambda_2\}$ over the 3-dimensional orthogonal group. We give the results for all 2-element partitions (λ_1, λ_2) of m for which $m \leqslant 12$. The corresponding reduction for the 3-dimensional rotation group is obtained from the results for the 3-dimensional orthogonal group by simply setting $\epsilon = 1$.

$\{1^2\} = \epsilon[1]$; $\{21\} = \epsilon[2] + [1]$;

$\{31\} = \epsilon[3] + [2] + \epsilon[1]$, $\{2^2\} = \{2\} = [2] + [0]$;

$\{41\} = \epsilon[4] + [3] + \epsilon[2] + [1]$, $\{32\} = \epsilon\{31\} = [3] + \epsilon[2] + [1]$;

$\{51\} = \epsilon[5] + [4] + \epsilon[3] + [2] + \epsilon[1]$, $\{42\} = [4] + \epsilon[3] + 2[2] + [0]$, $\{3^2\} = \epsilon\{3\} = \epsilon[3] + \epsilon[1]$;

$\{61\} = \epsilon[6] + [5] + \epsilon[4] + [3] + \epsilon[2] + [1]$, $\{52\} = [5] + \epsilon[4] + 2[3] + \epsilon[2] + [1]$, $\{43\} = \{41\}$;

$\{71\} = \epsilon[7] + [6] + \epsilon[5] + [4] + \epsilon[3] + [2] + \epsilon[1]$, $\{62\} = [6] + \epsilon[5] + 2[4] + \epsilon[3] + 2[2] + [0]$, $\{53\} = \epsilon\{52\}$, $\{4^2\} = \{4\}$;

$\{81\} = \epsilon[8] + [7] + \epsilon[6] + [5] + \epsilon[4] + [3] + \epsilon[2] + [1]$, $\{72\} = [7] + \epsilon[6] + 2[5] + \epsilon[4] + 2[3] + \epsilon[2] + [1]$, $\{63\} = \epsilon[6] + [5] + 2\epsilon[4] + 2[3] + \epsilon[2] + [1]$, $\{54\} = \epsilon\{51\}$;

$\{91\} = \epsilon[9] + [8] + \epsilon[7] + [6] + \epsilon[5] + [4] + \epsilon[3] + [2] + \epsilon[1]$, $\{82\} = [8] + \epsilon[7] + 2[6] + \epsilon[5] + 2[4] + \epsilon[3] + 2[2] + [0]$, $\{73\} = \epsilon[7] + [6] + 2\epsilon[5] + 2[4] + 2\epsilon[3] + [2] + \epsilon[1]$, $\{64\} = \{62\}$, $\{5^2\} = \epsilon\{5\}$;

$\{10.1\} = \epsilon[10] + [9] + \{81\}$, $\{92\} = [9] + \epsilon[8] + [7] + \{72\}$, $\{83\} = \epsilon[8] + [7] + 2\epsilon[6] + 2[5] + 2\epsilon[4] + 2[3] + \epsilon[2] + [1]$, $\{74\} = \epsilon\{73\}$, $\{65\} = \{61\}$;

$\{11.1\} = \epsilon[11] + [10] + \{91\}$, $\{10.2\} = [10] + \epsilon[9] + [8] + \{82\}$, $\{93\} = \epsilon[9] + [8] + 2\epsilon[7] + 2[6] + 2\epsilon[5] + 2[4] + 2\epsilon[3] + [2] + \epsilon[1]$, $\{84\} = [8] + \epsilon[7] + 2[6] + 2\epsilon[5] + 3[4] + \epsilon[3] + 2[2] + [0]$, $\{75\} = \epsilon\{72\}$, $\{6^2\} = \{6\}$.

For $n > 3$ the reduction of $\{\lambda_1, \lambda_2\}$ is the same for every value of n and is obtained by the method outlined above. The results for $m \leqslant 10$ are:

$\{1^2\} = [1^2]$; $\{21\} = [21] + [1]$; $\{31\} = [31] + [2] + [1^2]$, $\{2^2\} = [2^2] + [2] + [0]$;

$$\{41\} = [41] + [3] + [21] + [1], \{32\} = [32] + [3] + [21] + [1];$$
$$\{51\} = [51] + [4] + [31] + [2] + [1^2], \{42\} = [42] + [4] + [31] + [2^2] + 2[2] + [0], \{3^2\} = [3^2] + [31] + [1^2];$$
$$\{61\} = [61] + [5] + [41] + [3] + [21] + [1], \{52\} = [52] + [5] + [41] + [32] + 2[3] + [21] + [1], \{43\} = [43] + [41] + [32] + [3] + [21] + [1];$$
$$\{71\} = [71] + [6] + [51] + [4] + [31] + [2] + [1^2], \{62\} = [62] + [6] + [51] + [42] + 2[4] + [31] + [2^2] + 2[2] + [0], \{53\} = [53] + [51] + [42] + [3^2] + [4] + 2[31] + [2] + [1^2], \{4^2\} = [4^2] + [42] + [4] + [2^2] + [2] + [0];$$
$$\{81\} = [81] + [7] + [61] + [5] + [41] + [3] + [21] + [1], \{72\} = [72] + [7] + [61] + [52] + 2[5] + [41] + [32] + 2[3] + [21] + [1], \{63\} = [63] + [61] + [52] + [43] + [5] + 2[41] + [32] + 2[3] + [21] + [1], \{54\} = [54] + [52] + [43] + [5] + [41] + [32] + [3] + [21] + [1];$$
$$\{91\} = [91] + [8] + [71] + [6] + [51] + [4] + [31] + [2] + [1^2];$$
$$\{82\} = [82] + [8] + [71] + [62] + 2[6] + [51] + [42] + 2[4] + [31] + [2^2] + 2[2] + [0], \{73\} = [73] + [71] + [62] + [53] + [6] + 2[51] + [42] + [3^2] + 2[4] + 2[31] + [2] + [1^2], \{64\} = [64] + [62] + [53] + [4^2] + [6] + [51] + 2[42] + 2[4] + [31] + [2^2] + 2[2] + [0], \{5^2\} = [5^2] + [53] + [51] + [3^2] + [31] + [1^2].$$

Over the 4-dimensional rotation group each of the representations $[\lambda_1, \lambda_2]$ for which $\lambda_2 > 1$ is the sum of two irreducible representations of the same dimension but in all other cases these representations are irreducible not only over the orthogonal group but also over the rotation group.

For the 4-dimensional orthogonal group it remains only to discuss the representations $\{\lambda\}$ for which $(\lambda) = (\lambda_1, \lambda_2, \lambda_3)$ contains three non-zero parts. Of these, those for which $\lambda_1 = \lambda_2$ are cared for by means of the relation $\{\lambda_1, \lambda_2, \lambda_3\} = \epsilon^{\lambda_1}\{\lambda_1, \lambda_1 - \lambda_3, \lambda_1 - \lambda_2\}$ which is valid over the orthogonal subgroup. We give the reduction, over the 4-dimensional orthogonal group, for all 3-element partitions of $m \leqslant 10$. Note that, since the character of $[\lambda_1, \lambda_2]$ is zero over the reflection half of the group if $\lambda_2 > 0$, $\epsilon[\lambda_1, \lambda_2] = [\lambda_1, \lambda_2]$ when $\lambda_2 > 0$.

$$\{1^3\} = \epsilon[1]; \{21^2\} = \epsilon[2] + [1^2]; \{31^2\} = \epsilon[3] + [21] + \epsilon[1], \{2^21\} = \{21\} = [21] + [1];$$
$$\{41^2\} = \epsilon[4] + [31] + \epsilon[2] + [1^2], \{321\} = [31] + [2^2] + \epsilon[2] + [2] + [1^2], \{2^3\} = \{2\} = [2] + [0];$$
$$\{51^2\} = \epsilon[5] + [41] + \epsilon[3] + [21] + \epsilon[1], \{421\} = [41] + [32] + [3] + \epsilon[3] + 2[21] + [1], \{3^21\} = \{32\} = [32] + [3] + [21] + [1], \{32^2\} = \epsilon\{31^2\} = [3] + \epsilon[21] + [1];$$

$$\{61^2\} = \epsilon[6] + [51] + \epsilon[4] + [31] + \epsilon[2] + [1^2];\ \{521\} = [51] + [42] + [4] + \epsilon[4] + 2[31] + [2^2] + [2] + \epsilon[2] + [1^2],$$
$$\{431\} = [42] + [3^2] + \epsilon[4] + 2[31] + [2^2] + [2] + \epsilon[2] + [1^2],\ \{42^2\} = [4] + [31] + [2^2] + 2[2] + [0],\ \{3^22\} = \epsilon\{31\}$$
$$= [31] + \epsilon[2] + [1^2];$$
$$\{71^2\} = \epsilon[7] + [61] + \epsilon[5] + [41] + \epsilon[3] + [21] + \epsilon[1],\ \{621\} = [61] + [52] + [5] + \epsilon[5] + 2[41] + [32] + [3] + \epsilon[3] + 2[21] + [1],$$
$$\{531\} = [52] + [43] + \epsilon[5] + 2[41] + 2[32] + [3] + 2\epsilon[3] + 2[21] + \epsilon[1],\ \{52^2\} = [5] + [41] + [32] + 2[3] + [21] + [1],\ \{4^21\} = \{43\} = [43] + [41] + [32] + [3] + [21] + [1],\ \{432\} = \{421\},\ \{3^3\} = \epsilon\{3\} = \epsilon[3] + \epsilon[1];$$
$$\{81^2\} = \epsilon[8] + [71] + \epsilon[6] + [51] + \epsilon[4] + [31] + \epsilon[2] + [1^2],\ \{721\} = [71] + [62] + [6] + \epsilon[6] + 2[51] + [42] + [4] + \epsilon[4] + 2[31] + [2^2] + [2] + \epsilon[2] + [1^2],\ \{631\} = [62] + [53] + \epsilon[6] + 2[51] + 2[42] + [3^2] + [4] + 2\epsilon[4] + 3[31] + [2^2] + [2] + \epsilon[2] + [1^2],\ \{62^2\} = [6] + [51] + [42] + 2[4] + [31] + [2^2] + 2[2] + [0],\ \{541\} = [53] + [4^2] + [51] + 2[42] + [3^2] + [4] + \epsilon[4] + 2[31] + [2^2] + [2] + \epsilon[2] + [1^2],\ \{532\} = [51] + [42] + [3^2] + [4] + \epsilon[4] + 3[31] + [2^2] + [2] + \epsilon[2] + [1^2],\ \{4^22\} = \{42\},\ \{43^2\} = \{41^2\}.$$

For the 5-dimensional orthogonal group the reduction of the 2-element partition representations is the same as for the 4-dimensional orthogonal group (this reduction being now, however, the same for the rotation group as it is for the orthogonal group). We need only consider partitions $(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with not more than four non-zero parts since $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ is, over the orthogonal subgroup, the same as $\epsilon^{\lambda_5}\{\lambda_1 - \lambda_5, \lambda_2 - \lambda_5, \lambda_3 - \lambda_5, \lambda_4 - \lambda_5\}$. For the 4-element partition representations we have $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \epsilon^{\lambda_1}\{\lambda_1, \lambda_1 - \lambda_4, \lambda_1 - \lambda_3, \lambda_1 - \lambda_2\}$ so that, when $\lambda_1 = \lambda_2$ the reduction of the 4-element partition representations is furnished by the reduction of the 3-element partition representations. The reductions of the 3-element partition representations, for $m \leqslant 10$, are as follows:

$$\{1^3\} = \epsilon[1^2];\ \{21^2\} = \epsilon[21] + [1^2];\ \{31^2\} = \epsilon[31] + [21] + \epsilon[1^2],\ \{2^21\} = \epsilon[2^2] + [21] + [1];$$
$$\{41^2\} = \epsilon[41] + [31] + \epsilon[21] + [1^2],\ \{321\} = \epsilon[32] + [31] + [2^2] + \epsilon[21] + [2] + [1^2],\ \{2^3\} = [2^2] + [2] + [0];$$
$$\{51^2\} = \epsilon[51] + [41] + \epsilon[31] + [21] + \epsilon[1^2],\ \{421\} = \epsilon[42] + [41] + [32] + \epsilon[31] + \epsilon[2^2] + 2[21] + [3] + [1],\ \{3^21\} = \epsilon[3^2] + [32] + \epsilon[31] + [21] + \epsilon[1^2],\ \{32^2\} = [32] + \epsilon[2^2] + [3] + [21] + [1];$$
$$\{61^2\} = \epsilon[61] + [51] + \epsilon[41] + [31] + \epsilon[21] + [1^2],\ \{521\} = \epsilon[52] + [51] + [42] + \epsilon[41] + \epsilon[32] + [4] + 2[31] + [2^2] + \epsilon[21] +$$

$$[2] + [1^2],\ \{431\} = \epsilon[43] + [42] + [3^2] + \epsilon[41] + \epsilon[32] + 2[31] + [2^2] + \epsilon[21] + [2] + [1^2],\ \{42^2\} = [42] + \epsilon[32] + [4] + [31] + 2[2^2] + 2[2] + [0],\ \{3^22\} = [3^2] + \epsilon[32] + [31] + \epsilon[21] + [1^2];$$

$$\{71^2\} = \epsilon[71] + [61] + \epsilon[51] + [41] + \epsilon[31] + [21] + \epsilon[1^2],\ \{621\} = \epsilon[62] + [61] + [52] + \epsilon[51] + \epsilon[42] + [5] + 2[41] + [32] + \epsilon[31] + \epsilon[2^2] + [3] + 2[21] + [1],\ \{531\} = \epsilon[53] + [52] + [43] + \epsilon[51] + \epsilon[42] + \epsilon[3^2] + 2[41] + 2[32] + 2\epsilon[31] + [3] + 2[21] + \epsilon[1^2],\ \{52^2\} = [52] + \epsilon[42] + [5] + [41] + 2[32] + \epsilon[2^2] + 2[3] + [21] + [1],\ \{4^21\} = \epsilon[4^2] + [43] + \epsilon[42] + [41] + [32] + \epsilon[2^2] + [3] + [21] + [1];$$

$$\{81^2\} = \epsilon[81] + [71] + \epsilon[61] + [51] + \epsilon[41] + [31] + \epsilon[21] + [1^2],$$

$$\{721\} = \epsilon[72] + [71] + [62] + \epsilon[61] + \epsilon[52] + [6] + 2[51] + [42] + \epsilon[41] + \epsilon[32] + [4] + 2[31] + [2^2] + \epsilon[21] + [2] + [1^2],$$

$$\{631\} = \epsilon[63] + [62] + [53] + \epsilon[61] + \epsilon[52] + \epsilon[43] + 2[51] + 2[42] + [3^2] + 2\epsilon[41] + \epsilon[32] + [4] + 3[31] + [2^2] + \epsilon[21] + [2] + [1^3],$$

$$\{62^2\} = [62] + \epsilon[52] + [6] + [51] + 2[42] + \epsilon[32] + 2[4] + [31] + 2[2^2] + 2[2] + [0],$$

$$\{541\} = \epsilon[54] + [53] + [4^2] + \epsilon[52] + \epsilon[43] + [51] + 2[42] + [3^2] + \epsilon[41] + \epsilon[32] + [4] + 2[31] + [2^2] + \epsilon[21] + [2] + [1^2],$$

$$\{532\} = [53] + \epsilon[52] + \epsilon[43] + [51] + 2[42] + 2[3^2] + \epsilon[41] + 2\epsilon[32] + [4] + 3[31] + [2^2] + \epsilon[21] + [2] + [1^2],$$

$$\{4^22\} = [4^2] + \epsilon[43] + 2[42] + \epsilon[32] + [4] + [31] + 2[2^2] + 2[2] + [0],$$

$$\{43^2\} = \epsilon[43] + [3^2] + \epsilon[41] + \epsilon[32] + [31] + \epsilon[21] + [1^3].$$

The reductions, over the 5-dimensional orthogonal group, of the 4-element partition representations, for $m \leqslant 10$, are as follows:

$$\{1^4\} = \epsilon\{1\} = \epsilon[1];\ \{21^3\} = \epsilon[2] + \epsilon[1^2];$$

$$\{31^3\} = \epsilon[3] + \epsilon[21] + \epsilon[1],\ \{2^21^2\} = \{21^2\} = \epsilon[21] + [1^2];$$

$$\{41^3\} = \epsilon[4] + \epsilon[31] + \epsilon[2] + \epsilon[1^2],\ \{321^2\} = \epsilon[31] + \epsilon[2^2] + [21] + \epsilon[2] + \epsilon[1^2],\ \{2^31\} = [21] + [1];$$

$$\{51^3\} = \epsilon[5] + \epsilon[41] + \epsilon[3] + \epsilon[21] + \epsilon[1],\ \{421^2\} = \epsilon[41] + \epsilon[32] + [31] + 2\epsilon[21] + \epsilon[3] + [1^2],\ \{3^21^2\} = \epsilon[32] + [2^2] + \epsilon[3] + \epsilon[21] + \epsilon[1],\ \{32^21\} = \epsilon\{321^2\} = [31] + [2^2] + \epsilon[21] + [2] + [1^2],\ \{2^4\} = \{2\} = [2] + [0];$$

$$\{61^3\} = \epsilon[6] + \epsilon[51] + \epsilon[4] + \epsilon[31] + \epsilon[2] + \epsilon[1^2],\ \{521^2\} = \epsilon[51] + \epsilon[42] + [41] + \epsilon[4] + 2\epsilon[31] + \epsilon[2^2] + [21] + \epsilon[2] + \epsilon[1^2],\ \{431^2\} = \epsilon[42] + \epsilon[3^2] + [32] + \epsilon[4] + 2\epsilon[31] + \epsilon[2^2] + [21] + \epsilon[2] + \epsilon[1^2],\ \{42^21\} = [41] + [32] + \epsilon[31] + \epsilon[2^2] + [3] + 2[21] + [1],\ \{3^221\} = \{321\},\ \{32^3\} = \{31^3\};$$

$$\{71^3\} = \epsilon[7] + \epsilon[61] + \epsilon[5] + \epsilon[41] + \epsilon[3] + \epsilon[21] + \epsilon[1],$$

$$\{621^2\} = \epsilon[61] + \epsilon[52] + [51] + \epsilon[5] + 2\epsilon[41] + \epsilon[32] + [31] + \epsilon[3] + 2\epsilon[21] + [1^2],$$
$$\{531^2\} = \epsilon[52] + \epsilon[43] + [42] + \epsilon[5] + 2\epsilon[41] + 2\epsilon[32] + [31] + [2^2] + 2\epsilon[3] + 2\epsilon[21] + \epsilon[1],$$
$$\{52^21\} = [51] + [42] + \epsilon[41] + \epsilon[32] + [4] + 2[31] + [2^2] + \epsilon[21] + [2] + [1^2],$$
$$\{4^21^2\} = \{43^2\};\ \{4321\} = [42] + [3^2] + \epsilon[41] + 2\epsilon[32] + 2[31] + 2[2^2] + \epsilon[3] + 2\epsilon[21] + [2] + [1^2],$$
$$\{3^31\} = \{32\} = [32] + [3] + [21] + [1],\ \{42^3\} = [4] + [31] + [2^2] + 2[2] + [0],\ \{3^22^2\} = \{31^2\}$$

[1] Racah, G., *Rev. Mod. Phys.*, **11**, 494-496(1949).

[2] Littlewood, D. E., *The Theory of Group Characters*, Oxford Univ. Press, 1940, p 240.

THE HYPOTHESIS THAT ALL CLASSES ARE NAMEABLE

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In his book on the continuum hypothesis,[1] Gödel showed that if one adds to his axioms $A - D$ the additional statement V = L consistency is not impaired. We shall show that the further axiom

$$(A)(\exists x)(x \epsilon \omega . A = \phi x) \qquad \text{G1.}$$

causes no contradiction. Here ϕ is an operation definable in set-theory in such a way that G1 asserts that every class can be built up from E by the seven operations listed in B2–B8.

The operation ϕ is defined as follows: $\phi 2 = \text{E}$, $\phi(3^a.5^b) = \phi a \cdot \phi b$, $\phi(7^a) = D(\phi a)$, ..., $\phi(37^a) = Cnv_3(\phi a)$. $\phi n = 0$ if not defined by the above. We must show that ϕ and $\mathfrak{N}^*A$ meaning "A is nameable" are definable in set-theory in such a way that $\vdash \mathfrak{N}^*A \equiv (\exists n)(A = \phi n)$. (We write $\vdash p$ to mean that p is derivable from the axioms.) We define $\mathfrak{N}^*A$ as follows: Let P be the set of all n for which ϕn is defined. This set is evidently definable in set theory (and even recursive). Now we define $\mathfrak{N}^*A$ as

$$(\exists B)(\exists n)(n \epsilon \omega . (x)(x \epsilon B \supset (\exists w)(\exists z)(x = \langle w, z \rangle . z \leqslant n)).$$
$$(m)(m \leqslant n \supset ((m = 2.B''m = \text{E}) \vee (\exists a)(\exists b)(m \epsilon \text{P}.m = 3^a.5^b.B''m = B''a \cdot B''b) \vee (\exists a)(m \epsilon \text{P}.m = 7^a.B''m = D(B''a)) \vee \ldots \vee (\exists a)(m \epsilon \text{P}.m = 37^a.B''m = Cnv_3(B''a)) \vee (\sim m \epsilon \text{P}.B''m = 0))).B''n = A).$$

Now observe the following: (1) $\mathfrak{N}^*A$ has the form $(\exists B)\mathfrak{H}AB$, where $\mathfrak{H}$ is normal in Gödel's sense, i.e., contains no bound class variables. (2) B is the Cartesian sum of the classes $\phi 0$, $\phi 1$, $\phi 2$, ..., ϕn. (3) A (and incidentally also B) is uniquely determined given n, so we are justified in using the notation $A = \phi n$. (4) With a suitable defining postulate for ϕ, there is no difficulty in showing that $\vdash \mathfrak{N}^*A \equiv (\exists n)(n \epsilon \omega . A = \phi n)$. (5) By induction on n, we can easily show that the Cartesian sum B is nameable.

By observation (4), it suffices to show that $(A)\mathfrak{N}^*A$ is consistent relative to $A - D + (V = L)$.

THEOREM I. $\vdash (A \neq 0 . \mathfrak{N}^*A) \supset (\exists x)(x \epsilon A . \mathfrak{N}^*x)$.

Proof: First define an operation α singling out the element of least order from any class of constructible sets. (It is for this and this alone that we need the axiom V = L.) The "representative" αA is built up from A and E by the fundamental operations: hence $\vdash (A = 0 . \mathfrak{N}^*A) \supset (\alpha A \epsilon A . \mathfrak{N}^* \alpha A)$.

DEFINITIONS:

$(X_1^k) = (X_1) \ldots (X_k)$.

$\mathfrak{H}X_1^k = \mathfrak{H}X_1 \ldots X_k$. ($\mathfrak{H}$ is any notion.)

$\mathfrak{N}^*X_1^k = \mathfrak{N}^*X_1 . \ldots . \mathfrak{N}^*X_k$.

$[\ldots] =$ the result of restricting all bound variables of ... to $\mathfrak{N}^*$.

$\mathfrak{H}$ is absolute $= \vdash \mathfrak{N}^*X_1^k \supset (\mathfrak{H}X_1^k \equiv [\mathfrak{H}X_1^k])$.

$\mathfrak{H}$ is normal $= \mathfrak{H}X_1^k$ contains no bound class variables.

THEOREM II. *Every normal notion $\mathfrak{H}$ is absolute.*

Proof by induction on the number of logical operators occurring in $\mathfrak{H}X_1^k$. The basis and the inductive step for truth-functional composition are straightforward. Hence it suffices to show that if $\mathfrak{H}X_1^k x_{k+1}$ is absolute so is $(x_{k+1})\mathfrak{H}X_1^k x_{k+1}$, i.e., to derive

$$\{(x_{k+1})\mathfrak{H}X_1^k x_{k+1}\} \equiv (x_{k+1})(\mathfrak{N}^*x_{k+1} \supset [\mathfrak{H}X_1^k x_{k+1}])$$

from $\mathfrak{N}^*X_1^k$ and the absoluteness of $\mathfrak{H}$.

The implication from left to right is elementary. Assume then that the R.H.S. is true and the L.H.S. false. Since $\mathfrak{H}$ is normal, there exists the class A of all x for which $\mathfrak{H}X_1^k x_{k+1}$ fails; and by our assumption it $\neq 0$. By Theorem I it contains a nameable element y. Using our assumption again, $[\mathfrak{H}X_1^k y]$. By the absoluteness of $\mathfrak{H}$, $\mathfrak{H}X_1^k y$. But $y \epsilon A$, i.e., not $\mathfrak{H}X_1^k y$. Hence the equivalence holds and with it the theorem.

THEOREM III. *If $\vdash p$ then $\vdash [p]$.*

This is easily seen to follow from the case where p is an axiom. If p is an axiom of Group A, or is C1-3 or V = L, then p has the form $(X)\mathfrak{H}X$ with $\mathfrak{H}$ normal, so the theorem follows from theorem II. If p is B1-8, C4 or D1, the theorem follows from the fact that the classes asserted to

exist by these theorems are nameable. (In the case of D1 we need Theorem I.)

THEOREM IV. $\vdash \mathfrak{N}^*X \supset [\mathfrak{N}^*X]$.

Proof: $\mathfrak{N}^*X$ has the form $(\exists Y)\mathfrak{H}XY$, where $\mathfrak{H}$ is normal (Observation 1) and Y is nameable (Observation 5), so $\vdash \mathfrak{N}^*X \supset (\exists Y)(\mathfrak{N}^*Y.\mathfrak{H}XY) \supset (\exists Y)(\mathfrak{N}^*Y.[\mathfrak{H}XY])$ (Theorem II); i.e., $\vdash \mathfrak{N}^*X \supset [\mathfrak{N}^*X]$.

THEOREM V. *$(A)\mathfrak{N}^*A$ and hence G1 is consistent relative to $A - D +$ (V = L).*

Proof: Suppose if possible $\vdash (\exists A) \sim (\mathfrak{N}^*A)$. Then by Theorem III $\vdash [(\exists A) \sim (\mathfrak{N}^*A)]$ i.e., $\vdash (\exists A)(\mathfrak{N}^*A. \sim [\mathfrak{N}^*A])$. Then by Theorem IV $\vdash (\exists A)([\mathfrak{N}^*A]. \sim [\mathfrak{N}^*A])$; i.e., $A - D +$ (V = L) is inconsistent without $(A)\mathfrak{N}^*A$ if inconsistent with it.

One often hears it said that since there are indenumerably many sets and only denumerably many names, therefore there must be nameless sets. The above shows this argument to be fallacious.

[1] Gödel, K., *The Consistency of the Continuum Hypothesis*, Princeton, 1940.

ON CYCLOTOMY AND EXTENSIONS OF GAUSSIAN TYPE QUADRATIC RELATIONS INVOLVING NUMBERS OF SOLUTIONS OF CONDITIONAL EQUATIONS IN FINITE FIELDS

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Let p be an odd prime and m_1 and m_2 divisors of $p^n - 1$ and consider the relation

$$g^{r_1 + m_1 y_1} + g^{r_2 + m_2 y_2} = 1, \tag{1}$$

where g is a multiplicative generator of the non-zero elements of a finite ordered field of p^n, r_1 and r_2 are fixed integers. The number of different solutions of this relation in the y's with y_1 ranging over 0, 1, . . ., $m_1' - 1$ and y_2 ranging over the integers 0, 1, . . ., $m_2' - 1$ will be designated by

$$(r_1, r_2)_{m_1 m_2},$$

where

$$p^n - 1 = m_1 m_1' = m_2 m_2'.$$

A number of different types of quadratic relations involving $(r_1, r_2)_{m_1 m_2}$ have been given. Gauss[1] using such a relation was able actually to determine the value $(r_1, r_2)_{m_1 m_2}$ if $m_1 = m_2$ and m_1 is 3 or 4. His relations

were generalized by Lebesgue and Mitchell,[2] the latter stating his result in a form which is equivalent to the following:

$$\sum_{i=0}^{c-1} (j, i)_c\,(k - i, l - i)_c + A(j, k - l + \epsilon) = \sum_{i=0}^{c-1} (k, i)_c\,(j - i, l - i)_c + A(k, j - l + \epsilon)_c, \quad (2)$$

where

$$\frac{c^2}{m} A(r, s) = \sum_{a=0}^{c-1} \alpha^{ar} \cdot \sum_{b=0}^{c-1} \alpha^{bs}.$$

and where $\alpha = e^{2i\pi/c}$, $\epsilon = \text{ind } (-1)$, $g^{\text{ind } x} = x$ in $F(p)^n$, $(r_1, r_2)_{cc} = (r_1, r_2)_c$; $mc = p^n - 1$; j, k, and l arbitrary integers.

Mordell[3] by using a quite different method found another type of quadratic relation involving the number of solutions of

$$ax^n + by^k + c \equiv 0 \pmod{p}, \quad (3)$$

where a, b, and c are rational integers; and now x and y may represent *any* integers. These quadratic relations involve summations where a, b, and c range over the integers $0, 1, \ldots, p - 1$. He indicated an extension which involved the number of solutions of the congruence

$$c_1 x_1^{m_1} + c_2 x_2^{m_2} + \ldots + c_s x_s^{m_s} + c_{s+1} \equiv 0 \pmod{p}, \quad (4)$$

$s \geqq 2$, where the c's are given integers with $c_{s+1} \not\equiv 0$, modulo p. Again taking $n = 1$, Lebesgue, Dickson, and Hurwitz[4] gave quadratic relations of a still different character involving (r_1, r_2). The writer[5] obtained still other types of such quadratic relations. He, considering *only non-zero* solutions of (4), also extended the results of Hurwitz just referred to.

In the present paper, we shall modify and extend the method employed by Mitchell and his predecessors in obtaining (2). We shall then give various applications of the results,[6] especially to cyclotomy and modular relations involving binomial coefficients. The principal results are (14), (16), (19), (25), (27), (30), (31), and (38).

Suppose first that $m_1 = m_2 = c$, and let

$$p^n = 1 + cm. \quad (5)$$

Also, write $(i, j)_c$ for $(i, j)_{c,c}$. We have obviously

$$(i, j)_c = (j, i)_c = (i + kc, j + lc)_c, \quad (6)$$

for k and l any integers and using (1) after setting $r_1 = i$, $r_2 = j$, $m_1 = m_2 = m$ and dividing through by g^{i+mj}, we find[7]

$$(i, j)_c = (-i, j - i + \epsilon)_c, \quad (7)$$

where $\epsilon = (p^n - 1)/2$.

We will now generalize the above ideas. Consider the equation in $F(p^n)$

$$g^{r_1+m_1y_1} + g^{r_2+m_2y_2} + g^{r_3+m_3y_3} = 1, \tag{8}$$

the symbols being defined as in (1). Now write $E(r_i + m_iy_i)$ for $g^{r_i+m_iy_i}$. Hence, (6) can be written

$$E(r_1 + m_1y_1) + E(r_2 + m_2y_2) + E(r_3 + m_3y_3) = 1. \tag{9}$$

Let r_1, r_2, and r_3 be fixed and consider the possible values of y_1, y_2, and y_3. Let M be a positive integer divisible by both m_2 and m_3 and such that it is also a divisor of $p^n - 1$. Hence, there exists an i and a z such that

$$E(r_1 + m_1y_1) - 1 = E(i + Mz), \tag{10}$$

with i in the set $0, 1, \ldots, M - 1$; and z in the set $0, 1, \ldots, M' - 1$, with $MM' = p^n - 1$, unless, however, r_1 and y_1 are such that

$$E(r_1 + m_1y_1) - 1 = 0. \tag{11}$$

If this relation is satisfied, then (9) gives

$$E(r_2 + m_2y_2) + E(r_3 + m_3y_3) = 0. \tag{12}$$

With not both (11), (12) satisfied, then in order for (9) to be satisfied, we must have

$$E(i + Mz) + E(r_2 + m_2y_2) + E(r_3 + m_3y_3) = 0. \tag{13}$$

Consider (10) for a particular value of y_1 which yields a unique value of i, then we obtain from the last relation

$$1 + E(r_2 - i + m_2y_2') + E(r_3 - i + m_3y_3') = 0,$$

since m_2 and m_3 each divide M. Consequently, corresponding to the particular value of i used, there are $(r_2 - i + \epsilon, r_3 - i + \epsilon)$ values which satisfy this relation, where ϵ is the index of (-1) with respect to g; and there are $(r_1, i + \epsilon)$ values for y_1 and z which satisfy (10). Consequently, by taking all possible values of i, there are $\sum_{i=0}^{M-1} (r_2 - i + \epsilon, r_3 - i + \epsilon)(r_1, i + \epsilon)$ sets of solutions of (9) plus the number of solutions of (11) and (12). If the number of solutions of the latter set is $A(r_1, r_2, r_3)$, then the number of solutions of (9) is exactly

$$\sum_{i=0}^{M-1} (r_1, i)_{m_1M}(r_2 - i, r_3 - i)_{m_2m_3} + A(r_1, r_2, r_3),$$

since we can write $i + \epsilon$ for i, as we are taking i over the range for it which was mentioned. Since throughout the argument above we could have transposed r_1 and r_2, we can transpose r_1 and r_2 in the last expression, giving the relation, if M_1 is a multiple of m_1 and m_2 and a divisor of $p^n - 1$

$$\sum_{i=0}^{M-1} (r_1, i)_{m_1 M}(r_2 - i, r_3 - i)_{m_2 m_3} + A(r_1, r_2, r_3) =$$
$$\sum_{i=0}^{M_1-1} (r_2, i)_{m_2 M_1}(r_1 - i, r_3 - i)_{m_1 m_3} + A(r_2, r_1, r_3). \quad (14)$$

Let $[r_1, r_2, \ldots, r_s]$ denote the number of solutions of

$$E(r_1 + y_1 m_1) + \ldots + E(r_s + y_s m_s) = 0.$$

In lieu of (9), we may also consider the equation

$$E(r_1 + m_1 y_1) + E(r_2 + m_2 y_2) + E(r_3 + m_3 y_3) + E(r_4 + m_4 y_4) = 0. \quad (15)$$

Now use the same notation as before with the m's, and let N be a multiple of m_3 and m_4 with $p^n - 1 \equiv 0 \pmod{N}$ also N_1 a multiple of m_1 and m_4 with $p^n - 1 \equiv 0 \pmod{N_1}$; and carrying through a similar argument as employed in connection with (9), we obtain the relation

$$\sum_{i=0}^{N-1} [r_1, r_2, i]_{m_1 m_2 N}(r_3 - i, r_4 - i)_{m_3 m_4} + B(r_1, r_2, r_3, r_4) =$$
$$\sum_{i=0}^{N_1-1} [r_3, r_2, i]_{m_3 m_2 N_1}(r_1 - i, r_4 - i) + B(r_3, r_2, r_1, r_4), \quad (16)$$

where $B(r_1, r_2, r_3, r_4)$ is the product of the number of solutions of

$$E(r_1 + y_1 m_1) + E(r_2 + y_2 m_2) = 0, \quad (17)$$

by the number of solutions of

$$E(r_3 + y_3 m_3) + E(r_4 + y_4 m_4) = 0, \quad (18)$$

in the y's.

When we set all the m's equal in (14), we obtain the relation (2), given by Mitchell,[2] as it is easy to evaluate the A's in (14). In (2) put $j = b$ $k = d$, $l - k = b - a + \epsilon$, then (2) becomes, after using (7),

$$\sum_{i=0}^{c-1} (b, i)_c(b - a, i - d)_c + A(b, a - b) =$$
$$\sum_{i=0}^{c-1} (d, i)_c(d - a, i - b)_c + A(d, a - d). \quad (19)$$

We note also that $A(b, a - b) \neq 0$ if, and only if, $a \equiv 0$ and $b \equiv 0 \pmod{c}$ simultaneously, and the same for $A(d, a - d)$, with d replacing b.

By considering equations with a larger number of terms, we may find equations of any degree involving the parentheses symbols. We shall, in order to find a fairly simple relation indicating this, consider[8]

$$-1 + \sum_{k=1}^{4} E(r_k + m y_k) = 0, \quad (20)$$

where m is a divisor of $p^n - 1$; and we shall call the number of solutions

of this in the y's (r_1, r_2, r_3, r_4). The expression $-1 + E(r_1 + my_1)$ is either zero or $= E(i + mz)$ for some i and z. However, it cannot be zero if we assume that $r_1 \not\equiv 0 \pmod{m}$ which we shall do. Using our former methods, we have for N, the number of solutions of (20) in the y's.

$$N = \sum_{i=0}^{m-1} (r_1, i)(r_2 - i, r_3 - i, r_4 - i). \tag{21}$$

The second factor in each term here may be expressed as a product of two parentheses symbols by former methods; and we find if r_1', r_2', r_3' and r_4' is any permutation of r_1, r_2, r_3, r_4,

$$\sum_{i,j} (r_1, i)(r_2 - i, j)(r_3 - i - j, r_4 - i - j) = \sum_{i,j} (r_1', i)(r_2' - i, j)(r_3' - i - j, r_4' - i - j), \tag{22}$$

provided $r_1 \not\equiv 0$, $r_1' \not\equiv 0$, $r_4 - r_3 \not\equiv \epsilon \pmod{m}$, and $r_3' - r_4' \not\equiv \epsilon \pmod{m}$.

Also, we can see by induction that if we start with an equation in s terms, we may obtain by such methods an equation of degree $s - 1$ involving the parentheses symbols.

In (4) the m's were taken to be any divisors of $p^n - 1$. In particular, we can take any of the m's to be $p^n - 1$ itself. To illustrate how this idea may be used, we consider the equation

$$-g^a + E(r_1 + m_1y_1) + E(r_2 + m_2y_2) + E(r_3 + m_3y_3) = 0, \tag{23}$$

and consider the solutions in the y's of

$$-g^a + E(r_1 + m_1y_1) + g^i = 0,$$

or

$$-g^a + E(r_1 + m_1y_1) + E(i + (p^n - 1)z) = 0.$$

By the method used to obtain (14) we find that the number of solutions of (23) is with $r_1 \not\equiv 0 \pmod{m_1}$, $a = 0$,

$$\sum_{i=0}^{p^n-1} (r_1, i)_{m_1v}(r_2 - i, r_3 - i)_{m_2m_3},$$

where $v = p^n - 1$. As an application of this to the theory of power characters in a cyclotomic field, we can take our finite field as the residue classes modulo $\mathfrak{p}$ in the algebraic field defined by $\zeta = e^{2i\pi/m}$. Also, if $cm + 1 = p^n$, then write $g(\zeta)$ for the multiplicative generator of the non-zero elements of the residue classes modulo $\mathfrak{p}$ and also $(g(\zeta))^c \equiv \zeta \pmod{\mathfrak{p}}$ c even, m prime,

$$(g^i(\zeta) - 1)^c \equiv \left(\frac{g^i(\zeta) - 1}{\mathfrak{p}}\right) \pmod{\mathfrak{p}},$$

then if $r_1 \not\equiv 0 \pmod{m}$, $r_2 \not\equiv 0 \pmod{m}$,

$$\sum_i (r_2 - i, r_3 - i)_{mm_3} = \sum_j (r_1 - j, r_3 - j)_{mm_3}. \tag{24}$$

where i ranges over values in the set $1, 2, \ldots, p^n - 2$, such that

$$\left(\frac{g^i(\zeta) - 1}{\mathfrak{p}}\right) = \zeta^{r_1};$$

and j ranges over all the values in the same set such that

$$\left(\frac{g^j(\zeta) - 1}{\mathfrak{p}}\right) = \zeta^{r_2}.$$

Mitchell[2] (p. 172 relation (3)) found a formula which may be written as

$$\psi(\alpha^{s+t}\alpha^r)\psi(\alpha^s, \alpha^t) + (-1)^{ms} m \sum_{k=0}^{c-1} \alpha^{(s+t)k} =$$

$$\psi(\alpha^{r+t}, \alpha^s)\psi(\alpha^r, \alpha^t) + (-1)^{mr} m \sum_{k=0}^{c-1} \alpha^{(r+t)k}, \tag{25}$$

where $\alpha = e^{2i\pi/c}$, $p^n - 1 = cm$, and r, s, and t are arbitrary integers,[9]

$$\psi(\alpha^{\mu_1}, \alpha^{\mu_2}) = \sum_{h_1, h_2} \alpha^{\mu_1 h_1 + \mu_2 h_2} \tag{25a}$$

and

$$g^{h_1} + g^{h_2} = 1,$$

with both h_1 and h_2 ranging independently over the values $1, 2, \ldots, p^n - 2$. By specializing the values of r, s, and t, he finds the important known relation

$$\psi(\alpha^{\mu_1}, \alpha^{\mu_2})\psi(\alpha^{-\mu_1}, \alpha^{-\mu_2}) = p^n, \tag{26}$$

where $\mu_1 \not\equiv 0$, $\mu_2 \not\equiv 0$, $\mu_1 + \mu_2 \not\equiv 0 \pmod{c}$.

We now give a formula a bit different and slightly simpler than Mitchell's (25), which is also a generalization of (26). Multiply the equation (19) through by $\alpha^{as+br+dt}$ and then sum the resulting expressions when a, b, and d each range over $0, 1, \ldots, c-1$; we find, using Mitchell's method,

$$\sum_{i, a, b, d,} (b, i)_c(b - a, i - d)_c \alpha^{as+br+dt} + m \sum_{i=0}^{c-1} \alpha^{dt} =$$

$$\sum_{\ldots, a, d,} \alpha^{as+br+dt} (d, i)_c(d - a, i - b)_c + m \sum_{i=0}^{c-1} \alpha^{br}.$$

In view of the four summations, we can replace $i-d$ by d, and write a_1 in place of $b-a$ on the left, and similarly a_1 in place of $d-a$, and b_1 in place of $i-b$ on the right. This gives

$$\sum_{i,\,b}^{0,\,c-1}(b,\,i)\alpha^{bs+br+it}\sum_{a_1,\,d_1}^{0,\,c-1}(a_1,\,d_1)\alpha^{-a_1s-d_1t}+m\sum_{i=0}^{c-1}\alpha^{dt}=$$

$$\sum_{i,\,d}(d,\,i)\alpha^{d(s+t)+ri}\sum_{a_1,\,b_1}(a_1,\,b_1)\alpha^{-a_1s-b_1r}+m\sum_{i=0}^{c-1}\alpha^{br};$$

or using another formula,[10] we have

$$\psi(\alpha^{s+r},\,\alpha^{t})\psi(\alpha^{-s},\,\alpha^{-t})+m\sum_{d=0}^{c-1}\alpha^{dt}=\psi(\alpha^{s+t},\,\alpha^{r})\psi(\alpha^{-s},\,\alpha^{-r})+m\sum_{b=0}^{c-1}\alpha^{br}. \tag{27}$$

We then may state the

THEOREM: *If r, s, and t are arbitrary integers; $\psi(\alpha^{k},\,\alpha^{l})$ is defined in relation (25a); $p^{n}-1=cm$; then the relation (27) holds.*

Setting $r\equiv 0 \pmod{c}$ in (27) with $s\not\equiv 0 \pmod{c}$, $t\not\equiv 0 \pmod{c}$, and $s+t\not\equiv 0 \pmod{c}$, we obtain the relation (26).

In formulas (25) and (27) we used equal exponents in connection with our definition of the ψ number of (25a). Each was equal to c. However, if we wish to use two different types of roots of unity in (25a), we can achieve this by employing the latter relation and specializing the values of μ_1 and μ_2 therein. In particular, if we write in (25a)

$$\mu_1=k\mu_1',$$

where k is some divisor of c; and using the formula already referred to,[10] we have

$$\psi(\alpha^{k\mu_1'},\,\alpha^{\mu_2})=\sum_{i,\,j}(i,\,j)_c\alpha^{k\mu_1'i}\alpha^{\mu_2 j}$$

$$=\sum_{i,\,j}(i,\,j)_{k'c}\alpha^{k\mu_1'i}\alpha^{\mu_2 j}$$

where $kk'=c$, after using a formula given by the writer[8] (p. 49).

For applications of this formula we shall take $m=1$ in which case $c=p^{n}-1$. Let $\mathfrak{p}$ be an ideal prime divisor of p in the field defined by α selected so that $\alpha\equiv g \pmod{\mathfrak{p}}$, where now our finite field is represented by the residue classes modulo p, and g represents a primitive root modulo $\mathfrak{p}$ which corresponds exactly to the multiplicative generator of the non-zero elements of $F(p^{n})$, which we originally called g. Then, modulo $\mathfrak{p}$, we have by a known method, which has been employed in finding the ideal prime factors of the principal ideal (ψ),

$$\psi(\alpha^{s},\,\alpha^{t})\equiv\psi(g^{s},\,g^{t})$$
$$\equiv g^{\operatorname{ind} g^{hs}+t\operatorname{ind}(1-g^{h})}$$
$$\equiv g^{H},$$

where $H\equiv\operatorname{ind}(g^{hs}(1-g^{h})^{t})$ or

$$\psi(g^s, g^t) \equiv g^{hs}(1 - g^h)^t \pmod{\mathfrak{p}}.$$

Now assume $0 < s < v$, and $0 < t < v$, where $p^n - 1 = v$. We have

$$\sum_{h=0}^{p^n-2} g^{hs}(1 - g^h)^t \equiv \sum_{h=1}^{p^n-2} g^{hs}(1 - g^h)^t$$
$$\equiv \sum_{h=0}^{p^n-2} (-1)^j \binom{t}{j} g^{h(j+s)}.$$

Every term in the above is congruent to zero modulo $\mathfrak{p}$ unless $j + s = v$, where $p^n - 1 = v$; hence, $j = v - s$, and

$$\psi \equiv (-1)^{1+s}\binom{t}{v-s} \pmod{\mathfrak{p}}. \tag{28}$$

$s \not\equiv 0$, $t \not\equiv 0 \pmod{c}$; $0 < s < v$, and $0 < t < v$. Since we could have replaced s by t and t by s in our expression for ψ, it follows from the above that, modulo $\mathfrak{p}$, and since only rationals are involved, also modulo p,

$$(-1)^s\binom{s}{v-t} \equiv \binom{t}{v-s}(-1)^t.$$

Set

$$\binom{a}{b} = 0, \qquad b > a;$$

then putting $v - s$ for s and $v - t$ for t, we obtain[11]

$$(-1)^t\binom{v-s}{t} \equiv (-1)^s\binom{v-t}{s} \pmod{p} \tag{29}$$

for $v > t > 0$; $v > s > 0$.

Again, using (27) and (28), we find

$$\binom{|s+r|}{v-t}\binom{v-s}{t} \equiv \binom{|s+t|}{v-r}\binom{v-s}{r} \tag{30}$$

modulo $\mathfrak{p}$, and since only rational integers are involved, also modulo p. Here $|x|$ means the least positive or zero residue of x modulo v, $0 < i < v$, $i = r, s, t$; $|s + r| \neq 0$, $|s + t| \neq 0$. Proceeding in the same way with (25), we find[12] under the same conditions except that $|r + t|$ replaces $|s + r|$,

$$\binom{t}{v-s}\binom{|s+t|}{v-r} \equiv \binom{t}{v-r}\binom{|r+t|}{v-s}, \tag{31}$$

modulo p. The relation (31) may be directly generalized by employing the following theorem.[13] Let

$$\psi(\beta^{\mu_1}, \beta^{\mu_2}, \ldots, \beta^{\mu_s}) = \sum_{h_i} \prod_{i=1}^{s} \beta^{\mu_i h_i},$$

where h_i ranges independently over $1, 2, \ldots, p^n - 2$; $i = 1, 2, \ldots, s$, and where $\sum_{i=1}^{s} g^{h_i} = 1$, and $\beta = e^{2i\pi/v}$. Also, if $s > t$,

$$\psi'_{t,s} = \psi(\prod_{i=1}^{t} \beta^{\mu_i}, \beta^{\mu_{t+1}}, \ldots, \beta^{\mu_s}), \tag{32}$$

then

$$\psi_{1,s} = \psi_{1,2}\psi_{2,3}\cdots\psi_{s-1,s} \tag{33}$$

provided no one of the following relations hold

$$\begin{aligned} &\mu_i \equiv 0 \pmod{v};\ i = 1, 2, \ldots, s. \\ &\mu_1 + \ldots + \mu_j \equiv 0 \pmod{v};\ j = 2, 3, \ldots, s-1. \end{aligned} \tag{34}$$

In view of this result, suppose we consider the set of numbers $\mu_1, \mu_2, \ldots, \mu_s$ and take some other permutation of them, say $\mu_1', \mu_2', \ldots, \mu_s'$, and set up the function

$$\psi'_{t,s} = \psi(\prod_{i=1}^{t} \beta^{\mu' i}, \beta^{\mu' t+1}, \ldots, \beta^{\mu_s'}),$$

then

$$\psi_{1,s} = \psi'_{1,2}\psi'_{2,3}\cdots\psi'_{s-1,s} \tag{35}$$

provided no one of the following relations hold:

$$\begin{aligned} &\mu_i' \equiv 0 \pmod{v}, \qquad i = 1, 2, \ldots, s; \\ &\mu_1' + \ldots + \mu_j' \equiv 0 \pmod{v},\ j = 2, 3, \ldots, s-1. \end{aligned} \tag{36}$$

Hence (33) gives

$$\prod_{k=1}^{s-1} \psi_{k,k+1} = \prod_{k=1}^{s-1} \psi'_{k,k+1}. \tag{37}$$

We now employ the result in (28) on (37) in the same way that we obtained (31) from (25) and find if $|\mu_2| + |\mu_3| + \ldots + |\mu_s| = e$; $|\mu_2'| + |\mu_3'| + \ldots + |\mu_s'| = e'$,

$$(-1)^e \prod_{k=1}^{s-1} \binom{|\mu_1 + \ldots + \mu_k|}{v - |\mu_{k+1}|} \equiv (-1)^{e'} \prod_{k=1}^{s-1} \binom{|\mu_1' + \ldots + \mu_k'|}{v - |\mu'_{k+1}|} \tag{38}$$

modulo p, provided none of the congruences given in (34) and (36) hold, and $v = p^n - 1$. For $s = 3$, we obtain[14] (31). By employing certain transformations of (33), we may obtain many other congruences somewhat similar to (38), of which (30) is included as a special case. We hope to discuss these in another paper.

[1] *Werke*, **I**, 445–449; **II**, 67–92. Königlichen Gesellschaft der Wissenschaften zu Gottingen, 1870. He obtained his values in terms of definite quadratic forms.

[2] Lebesgue, V. I., *J. de Math.*, ser. 1, **3**, 113–145 (1838); Mitchell, H. H., *Trans. Am. Math. Soc.*, **17**, 165–177 (1916).

[3] *Math. Zeit.*, **37**, 193–209 (1933). As he was mainly interested in finding limits, he does not give the formulas we have referred to, explicitly.

[4] Lebesgue, V. I., *J. de Math.*, ser. 1, **19**, 289–296 (1854); Dickson, L. E., *J. für Math.*, **135**, 181–188 (1909); Hurwitz, A., *J. für Math.*, **136**, 272–292 (1909). Cf. also Pellet, *Bull. Soc. Math. France*, **15**, 80–93 (1886–87).

[5] These PROCEEDINGS, **35**, 681–685 (1949), and references cited there. These results were extended to other types of equations by Whiteman, *Ibid.*, **37**, 373–378 (1951), and quite general results obtained.

[6] The writer gave one application of (2) in which he showed that the number of solutions of any equation of the type (4) (but in $F(p^n)$ instead of only $F(p)$) with no x zero could be determined explicitly if we knew all the solutions of every such equation for $s = 2$ with $c_{s+1} \neq 0$.

[7] Mitchell, *loc. cit.*

[8] Cf. also these PROCEEDINGS, **32**, 47–52 (1946).

[9] Under the conditions $r + t \not\equiv 0 \pmod{c}$, $n = 1$, $s + t \not\equiv 0 \pmod{c}$, this formula is well known. Cf., for example, Weber, *Traité D'Algebre Superieure*, Gauthier Villars, 1898, p. 662.

[10] Vandiver, H. S., these PROCEEDINGS, **36**, 148 (1950), relation (15) as applied to relation (11), p. 147. To obtain another proof of (27) we might have taken the relation (25) and multiplied each side by $\psi(\alpha^{-s}, \alpha^{-t})\psi(\alpha^{-r}, \alpha^{-t})$ and applied (26) under the restrictions mentioned on the latter. However, to obtain (27) for all values of r, s, and t it appears necessary to consider a number of special cases which would make the complete proof of (27) by such a method more complicated than the one we have explained in full here. It is remarkable that both (25) and (27) hold for any r, s, and t.

[11] The relation (29) is true also for either r or s zero since it may be shown that

$$\binom{p^n - 1}{\mu} \equiv (-1)^\mu \pmod{p}.$$

Also, for $n = 1$ the "law of reciprocity" (29) was previously noted by the writer, these PROCEEDINGS, **35**, 337 (1949) in which special case the proof is immediate from the use of the explicit form for the binomial coefficients involved. The extension of this method to the case where $n > 1$ does not seem so obvious, but it may be proved by using a theorem due to Lucas, *Am. J. Math.*, **1**, 229–230 (1878).

[12] Cyclotomy has often been used to obtain congruences involving binomial coefficients and integers obtained by representing certain other integers as definite quadratic forms, for example, as in Stickelberger, *Math. Ann.*, **37**, 354–367 (1890); Davenport and Hasse, *J. für die Math.* (Crelle), **172**, 178–181 (1935). Special cases of these relations go back to Cauchy, Eisenstein and Jacobi. Relations (30), (31) and (38) of the present paper, involving binomial coefficients only, I have as yet been unable to find in the literature.

[13] This is a special case of a theorem given by O. B. Faircloth and the writer; these PROCEEDINGS, **36**, 265 (Theorem II) (1950).

[14] If the μ's employed in (38) are all different, then there is a congruence given by (38) for each permutation of the μ's, or in general $s!$ congruences. Unless the μ's are selected in a particular way, then both members of the congruence are zero because of the fact that one or more of the expressions in the lower line of the binomial coefficient symbols exceed the corresponding number in the upper line. To avoid this situa-

tion and also to satisfy the conditions on the μ's stated in the theorem, it seems necessary to select the μ's close to the modulus p. Take $s = 4$. Thus, for $p = 17$, if we select $\mu_1 = 5$, $\mu_2 = 6$, $\mu_3 = 7$, $\mu_4 = 8$, we find that $|\mu_1| + |\mu_2| + |\mu_3| = 2$ and $v - \mu_4 > 2$, whereas if we take $\mu_1 = 12$, $\mu_2 = 13$, $\mu_3 = 14$, and $\mu_4 = 15$, we find that each member of (38) reduces to 3 modulo 17. Also, for $p = 19$, $\mu_1 = 12$, $\mu_2 = 13$, $\mu_3 = 14$, $\mu_4 = 15$, each side reduces to 18 modulo 19. For $p = 23$, this set: $\mu_1 = 18$, $\mu_2 = 19$, $\mu_3 = 20$, $\mu_4 = 21$, is such that each side of (38) reduces to 19, mod 23. (In the above examples we happened to take the values of the μ's as consecutive, but obviously the conditions in the statement of our theorem do not require this.)

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*PRELIMINARY REPORT ON THE EXPERIMENTAL INDUCTION OF METASTASES FROM A HETEROLOGOUS CANCER GRAFT IN MICE**

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In our studies on the inflammatory response to foreign bodies and/or trauma we have utilized cortisone to inhibit normal responses to a variety of traumatic situations such as tuberculosis infection, wounds and irritative agents.[1-4] In continuing these investigations it has been our purpose to elicit information relative to the underlying physio-pathologic disturbances operative in circumventing the animal's normal defenses and therefore resulting in more extensive damage. We extended our studies to "living foreign bodies," i.e., cancer homoiografts.

Previous studies by Kaliss, Snell, Casey have shown that prior injection of mice [5-8] or rabbits[7] with lyophilized cancer or normal tissue from the strain indigenous to the cancer graft used enhanced the growth of the homoiograft in the strain so treated which ordinarily will not grow the graft. Subsequent joint studies between Kaliss and ourselves demonstrated that this phenomenon occurs with antiserums to the tissues prepared in either rabbits or mice[9,10] and is also true for grafts of normal spleen tissue in heterologous strains of mice[11] treated with lyophilized normal spleen. This afforded us a convenient test situation in which we postulated that: if the lyophilized tissue injections acted as a systemic trauma thus breaking down the host's ability to handle a second trauma, a live cancer graft; and if we could delay or inhibit such responses by further debilitating influences which specifically interfere with inflammatory responses, i.e., cortisone treatment; then we should be able to further

enhance the growth of the homoiograft. To this end the following experiments were conducted and are reported here in preliminary form because of the unusually striking results obtained.

Experimental Methods and Findings.—In the following studies the C57BL/6Jax strain of mice were used and the tumor Sarcoma I, an "A" strain tumor which originally arose in A strain mice and was grown by us in A/Cloudman mice.

Tumors were harvested from the A mice under aseptic conditions, lyophilized and resuspended in distilled water on a dry weight basis for injections. The live tumor grafts were fragments of Sarcoma I implanted by trocar in the supra-scapular region. Tumor growths were followed by palpation and by serial sacrifice and autopsy of mice in each group. Cortisone was administered subcutaneously in an aqueous suspension starting 0.5 mg. per mouse daily for 3 days and maintained by a daily dose (except Sundays) of 0.1 mg. per mouse thereafter.

Group I. Seventy mice cortisone maintained, injected with 3 doses 15 mg./mouse each of lyophilized Sarcoma I at 4-day intervals, followed by a live graft implant 1 week later. Cortisone was discontinued in 30 of these mice at the time of tumor graft and the remainder maintained on cortisone.

In both of these groups of mice the homoiograft grew slowly and in many began to become soft and necrotic. Fourteen days after tumor graft several animals exhibited distended abdomens and on sacrifice it was found that the abdomen contained ascitic fluid and extensive intraabdominal sarcomatosis involving mesentery, serosa of intestines, diaphragm, mesenteric nodes, pelvis, retroperitoneal tissues, peri-renal area, right kidney, liver. On subsequent days, the remaining animals showed similar metastases. Subsequent microscopic examination confirmed the gross finding and revealed sarcomata in other organs, e.g., pancreas. The detailed findings and analysis of the data will be reported at a later date when complete serial section studies are made along with data on mice to be sacrificed prior to tumor graft and daily after implantation.

Group II. Control. Twenty-seven mice treated with lyophilized Sarcoma I as above and injected daily with the suspending diluent used for cortisone suspension. Tumor implanted as above. No evidence of metastases was noted. The primary graft was growing progressively in all mice at time of sacrifice. Sacrifices were made at the same time to parallel group I.

Group III. Controls. Thirty mice treated with cortisone only and Sarcoma I implanted as above. Primary tumor was regressing in all mice and at autopsy the primary graft was necrotic and no gross evidence of spread noted.

Group IV and V. Controls. Sarcoma I implanted in 20 C57BL/6Jax and in the indigenous strain, 20 A/Cloudman mice. The graft regressed in all

the C57BL/6Jax and grew progressively in the A/Cloudman strain. No evidences of abnormalities were found in any of these mice at autopsy.

In another series of studies the above experimental conditions were duplicated in adrenalectomized mice. At this time the following preliminary findings are in hand:

1. Adrenalectomized mice treated as group I above do not show any gross evidence of metastases of the heterologous Sarcoma I.

2. Lyophilized-tissue-treated adrenalectomized mice grow the Sarcoma I implant whether treated with cortisone or not. If anything, adrenalectomy further enhances the Sarcoma I implant.

3. Untreated adrenalectomized mice do not grow the Sarcoma I.

To test the viability of the heterologous metastatic sarcoma fragments of the tissue were taken from the mesentery of two of the mice at autopsy and implanted into 6 normals of each of the following strains: C3H/Jax, C57BL/6Jax, C/Scott, Street, DBA and A/Cloudman. The grafts were palpable in *all six of these strains* within 4 days and at 7 days were all larger than 2 cm. on palpation. In the Street strain the grafts were over 3–4 cm. All six strains were sacrificed on the 9th day and the tumor and organs fixed for microscopic histo-pathological study. A duplicate set of the above strains similarly implanted are currently under observation for ultimate fate of the graft. It appears that the metastatic sarcomata may have lost their strain specific character. This remains to be established by more critically controlled experiments. Kaliss[12] is currently observing a similar change in strain specific character of a homoiograft from lyophilized tissue-enhanced mice in untreated mouse strains to which the tumor is not indigenous.

A parallel series of studies is under way in which wounding is used as a means of traumatic stimulus in addition to an homologous cancer graft. In these studies as in the above, it is intended to elicit information concerning the influence of the state of "reacting to trauma" on the graft. The preliminary experiments involve two situations: a rapid- and a slow-growing implanted tumor.

The rapidly-growing tumor is the Sarcoma S621 in C/Scott mice, the slow-growing tumor is the BW1898 in C57BL/6Jax mice. Each test set-up is comprised of 3 groups: control implant in normal mice; graft implant intramuscularly and tangent to a surgically induced wound on the back involving skin, subcutaneous fascia and muscle, the implant made 4 hours after wounding and the wound allowed to heal; graft implant intramuscularly and tangent to a wound as above, but the wound surgically opened daily in order to keep a situation of continuous "granulating wound."

The Sarcoma S621 in the control (not wounded) C/Scott mice grew progressively and all animals were dead with large tumors which had broken through the muscle wall and were growing intraabdominally in 11

to 19 days. This is average for this tumor/strain. The wounded C/Scott mice in whom the wounds were permitted to heal differed slightly from the controls: the graft grew more slowly and in $^1/_2$ the animals was not palpable until the 11th–15th day; deaths with tumor occurred between the 19th and 26th day. The mice in whom the wounds were surgically opened daily were considerably slower in growth; the first deaths[5] with small tumors occurring on the 20th day; the remaining 10 mice in this group were sacrificed on that day for comparative autopsy. It was found in this latter group that the tumor was smaller, well encapsulated, necrotic and did not break through the dorsal muscle wall into the thorax and abdomen. The tumors in the control mice had broken through and extended growth into the abdomen and showed moderate central necrosis. It appears that the maintenance of an active inflammatory response in the local tumor site acts to "contain" an homologous cancer graft which grows more slowly than in normal mice.

The experiments on the slow-growing graft are still in progress. It appears at this writing that the same results are being obtained.

A recent report[12] of cortisone-induced metastases of an homologous cancer graft in mice adds further to the data we are obtaining in support of our hypothesized physio-pathologic mechanism of action. These latter investigators, however, attribute their findings to a "tumorigenic" action of the cortisone despite their own finding that cortisone delayed and arrested the cancer implant. In addition they used a strain of mice (C3H) known to possess the mammary milk factor with a mammary adenocarcinoma and this may play a role in their contradictory findings. It is beyond doubt, however, that growth of intraabdominal sarcomata was induced and it remains now to establish the mechanism by further experimentation.

It is interesting that we were able to predict our results based on our tentative hypothesis of mechanism knowing full well that a heterologous cancer graft has never been made to metastasize.

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AN INVARIANT FORCE FUNCTION IN MUSCLE ACTIVITY

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When a single nerve impulse reaches a neuromuscular junction it initiates a chain of events culminating in the characteristic twitch response of the muscle fiber. If the fiber length is fixed during activity the tension rises to a maximum and declines gradually to the resting level. The so-called isometric twitch is a basic all or none increment of activity in the sense that its magnitude and time course are independent of the nature of the stimulus. All suprathreshold electric current stimuli elicit more or less identical twitches regardless of current duration or rate of change of current. Any continued maintenance of actively developed tension is possible under normal conditions only in the event of repetitive stimulation of the fiber. A progressive summation of staggered twitches then suffices to maintain tension above the resting level. A fundamental point to emphasize is that muscle activity is manifest either as a twitch or as a sum of twitches.

However, the magnitude of the isometric twitch can be altered by changing certain variables. As the length of the fiber is increased, the maximum change in tension during a twitch is reduced, but the time parameters of tension change are unaltered in an unfatigued fiber. This at once suggests that some invariant process may underlie each isometric twitch regardless of extent to which the tension may change.

However, isotonic activity of muscle involves further complications. If a muscle shortens against a constant load the time course of the corresponding twitch is not completely independent of the load. It is the purpose of this investigation to disclose a unique all-or-none time function which is invariant for all muscle twitches carried out isotonically. By relating the isotonic peak twitch deflection to the derivative of length with respect to tension, it can be shown that a constant force is generated within the muscle at the time of the twitch deflection maximum regardless of existing tension or length. In so far as the time course of the isotonic

twitch is independent of length or tension it may be stated that an invariant force-time function underlies the mechanism of the isotonic twitch.

In our experiments turtle neck retractor muscles about 10 cm. in length were allowed to contract against a weighted lever whose effective mass was placed as near the fulcrum as possible. Twitches elicited by supramaximal stimuli were recorded kymographically. Representative twitches involving different loads on the muscle are shown in figure 1 for a typical experiment. As the load is increased, the resting length of the muscle is of course increased, but the twitch deflections are greatly reduced in magnitude. Any pair of twitches can be very nearly superimposed during the contraction phase if the ordinates of one are multiplied by the proper constant. The three lower displacement-time curves of figure 1 were so

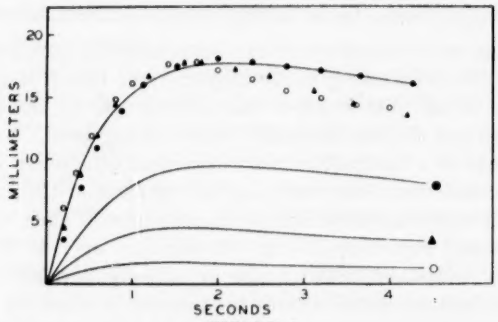

FIGURE 1

Single twitches of a 12 cm. turtle neck retractor muscle at 8 C. Stimuli are supramaximal. The four continuous curves indicate displacement-time functions corresponding to four different loads on the muscle of 5, 12, 60 and 120 grams. Symbols along the uppermost curve indicate ordinates of the corresponding curves below multiplied by a constant in each case.

adjusted with the uppermost curve taken as the comparison standard. For this set of curves the time parameters of the shortening phase are very nearly independent of the load. However, the delayed relaxation exhibits some qualities of "stickiness" in the sense that as the load decreases to very small values, the muscle, once contracted, does not completely relax to its original length. One may then provisionally assume that "stickiness" can very reasonably obscure any continuous invariant process underlying the twitch throughout its entirety. For purposes of further analysis then, only the contraction phase of the twitch will be taken into consideration.

The derivative of length with respect to tension was approximated by measuring the initial shortening ΔL of the resting loaded muscle as a small constant weight Δw was suddenly removed from the load. The increment

ΔL and peak twitch deflection are both plotted as a function of the load in figure 2. It is at once obvious that twitch height is preportional to ΔL and therefore to $\Delta L/\Delta w$ throughout a wide range of change in each variable.

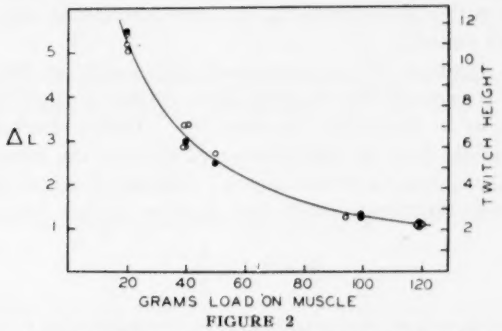

FIGURE 2

The ordinate ΔL represents millimeters change in length of a resting muscle on suddenly removing 15 grams from the initial load on the muscle. The ordinate designated as twitch height refers to maximum change in length in millimeters of the muscle following stimulation by means of a single supramaximal shock.

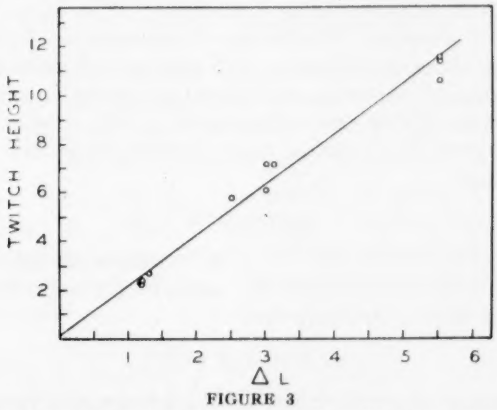

FIGURE 3

Twitch heights of figure 2 are plotted against ΔL in millimeters. Any point on the regression line corresponds to a unique load on the muscle.

The proportionality is further emphasized by the regression line shown in figure 3. Therefore, peak twitch deflection y_m is related to $\Delta L/\Delta w$ by an equation of the form

$$y_m = k \frac{\Delta L}{\Delta w} \qquad (1)$$

in which the constant k has the dimensions of a force. Total tension is constant throughout the system in this slowly acting muscle, but reorganization of the system during activity may be thought of as resulting in a redistribution of force components such that one of them, associated in particular with active shortening, becomes manifest in determining the constant k of equation (1).

However, equation (1) as formulated applies only at time of twitch maximum. It remains then to determine whether k might not possibly be replaced by a force-time function $f(t)$. Twitch displacement-time parameters would then be independent of load on the muscle. Hence, in so far as the adjusted curves of figure 1 coincide, it is possible to formulate a force time function $f(t)$ such that equation (1) may be replaced by

$$y = Kf(t)\frac{\Delta L}{\Delta w} \tag{2}$$

where y is displacement at any time during the twitch and K is a constant. The function $f(t)$ thus appears as the desired invariant time function underlying all isotonic twitches.

Additional significance may be attached to equation (2) by reference to the recent work of Buchtal & Kaiser[1] on kinetic elasticity of muscle and fixation of chain linkages during contraction. It is assumed that the muscle fiber is composed of molecular "transmutation chains" characterized by two states of equilibrium with different individual link lengths. Any single link may exist either as an elongated β element or as a contracted α element which may or may not be fixated in the α_f state. As an α_f link the contracted element is not free to be transformed into the β state. The equilibrium

$$\alpha_f \leftrightharpoons \alpha_1 \leftrightharpoons \beta \tag{3}$$

may then be formulated in which α_f is the fraction of the links in the fixed form, α_1 the fraction in contracted but unfixed form and β the fraction in the elongated state. Obviously then

$$\alpha_f + \alpha_1 + \beta = 1. \tag{4}$$

The rate of change of β with respect to time is assumed to yield

$$\frac{d\beta}{dt} = \nu(T)e^{-A/kT}[\alpha_1 e^{P\lambda/kT} - \beta e^{-P\lambda/kT}] \tag{5}$$

where A is the activation energy; P, the load on the chain; λ, the difference in length between either the α or β equilibrium position (potential minimum) and the peak of the potential barrier separating them in the loaded state; T, the absolute temperature; k, a proportionality constant and $\nu(T)$ is a factor giving frequency of thermal collisions with due considera-

tion to the spatial possibilities of transmutation. On substitution of W for $\nu(T)e^{-A/kT}$ and p for $P\lambda/kT$, the rate of change of α_1, with respect to time is given by

$$\frac{d\alpha_1}{dt} = W[\beta e^{-p} - \alpha_1 e^{p}] + D\alpha_f - FC\alpha_1 \qquad (6)$$

where D and F are proportionality factors and C is the concentration of a link-fixating substance released on stimulation. Similarly

$$\frac{d\alpha_f}{dt} = FC\alpha_1 - D\alpha_f \qquad (7)$$

The length of the chain per link is assumed to be

$$L = 1 + 3\beta \qquad (8)$$

on the basis of difference in length between a maximally contracted muscle fiber and a heavily loaded resting fiber. When $d\beta/dt = C = 0$ it follows from equations (5) and (8) that

$$L = 1 + \frac{3}{1 + e^{-p}} \qquad (9)$$

Now from equations (5), (6) and (7) there can be derived the second order differential equation

$$\frac{d^2\beta}{dt^2} + [W(e^{p} + e^{-p}) + FC + D]\frac{d\beta}{dt} + [F_vCWe^{-p} + DW(e^{p} + e^{-p})]\beta - DWe^{-p} = 0 \qquad (10)$$

It is assumed that on repetitive stimulation C is released instantaneously and maintained as a constant quantity. From equation (8) and the integrated form of equation (10) in which one exponential time constant is neglected there can be obtained the approximate relation for maximum shortening velocity

$$\left(\frac{dL}{dt}\right)_m \approx \frac{3FC}{(e^{p} + e^{-p})^2} \qquad (11)$$

Equations (9) and (11) are to be analyzed in relation to the findings of this investigation. From equation (2) on differentiation we obtain for the maximum velocity of shortening

$$\left(\frac{dy}{dt}\right)_m = Kf'(t_m)\frac{\Delta L}{\Delta w} = \left(\frac{dL}{dt}\right)_m \qquad (12)$$

where $f(t_m)$ is, of course, a constant for all twitches, namely, the value of $f(t)$ at the time of maximum rate of shortening. Now since the first few

twitches summating to produce a tetanus are essentially similar in temporal configuration and constitute approximately equal increments of shortening, one is justified in applying equation (12) during the very early phase of tetanic contraction when velocity of shortening is also at a maximum. The term $Kf(t_m)$ will then represent a force component due to addition of several twitches. Now on substitution of equations (9) and (11) in (12) there is obtained

$$\frac{3FC}{(e^{2p} + e^{-2p} + 2)} = \frac{6Kf'(t_m)}{(e^{2p} + e^{-2p} + 2)} \tag{13}$$

where $Kf'(t_m) = FC/2$ and C represents the concentration of link fixating substance.[1] These results of course imply that the invariancy of $f(t)$ in equation (2) is dependent on the release of a constant amount of fixating substance per twitch, regardless of muscle fiber length or tension.

However, certain complications arise if the concentration of the link fixator C is assumed to be a stepwise function of the time as Buchtal & Kaiser have done.[1] Integration of equation (10) then leads to a sum of 2 exponential terms whose time constants are functions of the load. However, the time parameters of the twitch contraction phase appear to be independent of the load in so far as the adjusted curves of figure 1 may be considered as coincident. This would imply then that the concentration C of the fixator in equation (10) should be represented as $C(t)$, a time function whose time parameters are very nearly those of the twitch itself. A plausible configuration of the $C(t)$ function can be derived very simply from considerations of first order reactions.

In the system

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C \xrightarrow{k_3} D \xrightarrow{k_4} \tag{14}$$

if A is suddenly released in a manner described by the function $A_0e^{-k_1t}$, where A_0 is the initial concentration of A, B will follow a time course described by the sum of two exponential terms; C, by a sum of three exponential terms; D, by four, etc. As Gilson *et al.*[2] and Schoepfle *et al.*[3] have shown, a set of values for the k's in (14) can be chosen such that a sum of three exponential terms describing C as a function of time will quite closely coincide with the isometric twitch tension time course. A function of this sort might well replace C in equation (10). It is recognized that diffusion of the link fixator to or from its site of action might well account in part for the configuration of $C(t)$.

Mention should also be made of the role of various structures which might act in parallel with the fibrils of "chains" actually responsible for contraction. Muscle connective tissue, inactive fibrils or the sarcolemma itself might be considered non-linear or non-Hookean springs acting in parallel with the invariant force $f(t)$ of equation (2).

[1] Buchtal, Fritz, and Kaiser, E., "The Rheology of the Cross Striated Muscle Fiber," *Dan. Biol. Medd., Copenhagen*, **21**, no. 7 (1951).

[2] Gilson, A. S., Walker, S. M., and Schoepfle, G. M., *J. Cell. Comp. Physiol.*, **24**, 185–199 (1944).

[3] Schoepfle, G. M., and Gilson, A. S., *Ibid.*, **26**, 119–130 (1945).

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